Homological Stability for Hurwitz Spaces

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Kurzzusammenfassung

***Schlagworte*: homologische Stabilität, verzweigte Überlagerungen, Zopfgruppen**

In der vorliegenden Arbeit beschäftigen wir uns mit Hurwitzräumen im Hinblick auf homologische Stabilisierung. In den vergangenen Jahrzehnten hat sich die Untersuchung homologischer Stabilitätseigenschaften zu einem ergiebigen Forschungsfeld besonders innerhalb der Topologie entwickelt. Unter einem Hurwitzraum verstehen wir einen Modulraum verzweigter, nicht notwendig zusammenhängender Überlagerungen einer Kreisscheibe mit festgelegter Strukturgruppe und Anzahl von Verzweigungspunkten. Solche Räume sind im Allgemeinen nicht zusammenhängend, bilden allerdings eine unverzweigte Überlagerung eines Konfigurationsraums. Wir wählen eine Folge von Unterräumen von Hurwitzräumen, die sich für unsere Untersuchungen eignet.

Wir verallgemeinern ein Resultat von Ellenberg, Venkatesh und Westerland, indem wir zeigen, dass die homologische Stabilisierung unserer Folge von Hurwitzräumen nur von gewissen Eigenschaften ihrer nullten Homologiegruppen abhängt. Aus diesem Kriterium leiten wir konkrete Fälle ab, in denen homologische Stabilität vorliegt. Um zu unserem Resultat zu gelangen, definieren und untersuchen wir eine neue Klasse hochzusammenhängender Simplizialkomplexe.

Abstract

***Keywords*: homological stability, branched coverings, braid groups**

In the present thesis, we study Hurwitz spaces with regard to homological stabilization. In the last decades, the investigation of homological stability questions has evolved into a fruitful area of research, especially within topology. By a Hurwitz space, we mean a moduli space of branched, not necessarily connected coverings of a disk with fixed structure group and number of branch points. In general, such spaces are disconnected, but they form an unbranched covering space of a configuration space. We choose a sequence of subspaces of Hurwitz spaces which is suitable for our investigations.

We generalize a result by Ellenberg, Venkatesh, and Westerland by showing that homological stabilization of our sequence of Hurwitz spaces depends only on certain properties of their zeroth homology groups. From this criterion, we deduce specific cases in which homological stability holds. In order to arrive at our result, we define and study a new class of highly connected simplicial complexes.

Contents

1. Introduction

Understanding the topology of moduli spaces is a key aspect in order to grasp the behavior in families of the parametrized objects. Now, moduli spaces often come in sequences, such as the moduli spaces $\mathcal{M}_g$ of Riemann surfaces of genus g. In some cases, such sequences satisfy *homological stability*. Indeed, by [Har85], the homology groups $H_p(\mathcal{M}_g;\mathbb{Q})$ are independent of g in a range of dimensions growing with g.

In recent years, the study of (co-)homological stability phenomena has been of great interest in algebraic and geometric topology as well as algebraic geometry. Classical results include stabilization for the group homology of the symmetric groups $\mathcal{S}_n$ ([Nak60]), the general linear groups GL_n ([Maa79], [vdK80]), and the Artin braid groups Br_n ([Arn70]). The theorem for braid groups builds a bridge to moduli spaces: Br_n is classified by the configuration space Conf_n, which parametrizes subsets of $\mathbb{C}$ of size n. Hence, the sequence $\{\mathrm{Conf}_n\}$ is homologically stable.

In fact, several homological stability theorems are concerned with classifying spaces of groups G_n. There is by now a standard approach (cf. [HW10]) to the proof of such results which involves the construction of a highly connected simplicial complex with a G_n-action and the investigation of the associated spectral sequence.

Hurwitz spaces as moduli spaces of branched covers of $\mathbb{C}$ (or $\mathbb{P}^1_{\mathbb{C}}$) came up in the second half of the 18th century in the work of Hurwitz ([Hur91]). Their construction helped proving the connectivity of $\mathcal{M}_g$ in [Sev68], and they play an important role in arithmetic applications such as the Regular Inverse Galois Problem (cf. [FV91]).

In this thesis, we are interested in homological stabilization for Hurwitz spaces. It is worth mentioning the proximity of these spaces to both moduli spaces of curves and configuration spaces: The total space of a branched covering is a Riemann surface, whereas the branch locus defines an element of a configuration space. Having the homological stability theorems for both $\mathcal{M}_g$ and Conf_n in mind, it seems worthwhile to study Hurwitz spaces in this direction.

In fact, a specific class of Hurwitz spaces was shown to satisfy homological stability in [EVW16]. In this thesis, we follow this path in order to broaden the class of sequences of Hurwitz spaces which are known to be homologically stable.

1.1. Summary of the results

The central result of this thesis is Theorem 1 which extends the results from [EVW16] to a more general setting. By Theorem 5 and further results from Chapter 4, *colored plant complexes* provide most of the technical machinery needed for this generalization.

Let G be a finite group, $c = (c_1, \dots, c_t)$ a tuple of t distinct conjugacy classes in G, $\underline{\xi} \in \mathbb{N}_0^t$, and $n \in \mathbb{N}_0$. Furthermore, let D be a closed disk and $* \in \partial D$ a marked point.

This thesis is about the homology of the Hurwitz spaces $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$. These spaces parametrize isomorphism classes of *marked* $n \cdot (\sum_{i=1}^t \xi_i)$*-branched* G*-covers of* D *with coloring* $\underline{\xi}$. Such covers are given by a branch locus S of cardinality n in the interior of D and an unbranched (not necessarily connected) cover $p\colon Y \to D \setminus S$ such that there is a unique identification of the fiber above $*$ with G. In addition, we demand that $n\xi_i$ of the local monodromies around the branch points lie in c_i, for $i = 1, \dots, t$.

The group G acts on $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ by changing the unique identification with G of the fiber above $*$. The quotient of this action is denoted by $\mathcal{H}^c_{G,n\cdot\underline{\xi}}$.

Our central result is an algebraic criterion for the existence of a homological stability theorem for the sequence $\{\mathrm{Hur}^c_{G,n\cdot\underline{\xi}} \mid n \geq 0\}$. We prove that homological stability for these spaces is encoded in the zeroth homology modules. For a commutative ring A, there is a graded ring structure on

$$R = \bigoplus_{n \geq 0} H_0(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; A).$$

Properties of this *ring of connected components* are key to our theorem.

Theorem 1 (Theorem 5.7, Theorem 5.35, Corollary 5.5)
Let A be a commutative ring. Assume there is a central homogeneous element $U \in R$ of degree k such that the multiplication $R \xrightarrow{U\cdot} R$ is an isomorphism in high degrees. Then there are constants a, b (depending on U, G, c, $\underline{\xi}$, and A) such that for all $p \geq 0$, there is an isomorphism

$$H_p(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; A) \cong H_p(\mathrm{Hur}^c_{G,(n+k)\cdot\underline{\xi}}; A)$$

in the stable range $n > ap + b$.

If A is a field whose characteristic is either zero or prime to $|G|$, and if U is invariant under the G-action on R, there is another isomorphism in the same range:

$$H_p(\mathcal{H}^c_{G,n\cdot\underline{\xi}}; A) \cong H_p(\mathcal{H}^c_{G,(n+k)\cdot\underline{\xi}}; A).$$

In the theorems, we provide an explicit stable range in terms of characteristics of the element U and the map $R \xrightarrow{U\cdot} R$.

The *colored configuration space* $\mathrm{Conf}_{n\cdot\underline{\xi}}$ parametrizes unordered configurations of $n \cdot (\sum_{i=1}^{t} \xi_i)$ points in the interior of D, where for all $i = 1, \dots, t$, exactly $n\xi_i$ points are labeled with the color i. The relevance of these spaces to our thesis lies in the fact that $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ admits an unbranched covering map to $\mathrm{Conf}_{n\cdot\underline{\xi}}$.

Using Theorem 1 and a result from [EVW12], we conclude homological stability for Hurwitz spaces parametrizing connected covers, including an identification of the rational stable homology:

Theorem 2 (Theorem 6.20)
If for $n \geq 1$, all covers in $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ are connected, there is a central homogeneous element $U \in R$ and constants a, b as in Theorem 1 such that for any $p \geq 0$, there are isomorphisms

$$H_p(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; \mathbb{Z}) \cong H_p(\mathrm{Hur}^c_{G,(n+1)\cdot\underline{\xi}}; \mathbb{Z})$$

in the stable range $n > ap + b$. If s is the stable number of connected components of $\mathrm{Hur}^c_{G,N\cdot\underline{\xi}}$ for $N \gg 0$, we have

$$H_p(\mathrm{Hur}^c_{n\cdot\underline{\xi}}; \mathbb{Q}) \cong H_p(\mathrm{Conf}_{n\cdot\underline{\xi}}; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}^s$$

in the same range $n > ap + b$.

Many of the examples in this thesis deal with the case where $G = \mathcal{S}_3$ is the symmetric group on three elements. Apart from the *diagonal* homological stability result which can be deduced from Theorem 2 (Example 6.21) we prove homological stabilization in a *vertical* direction:

Theorem 3 (Theorem 6.13)
Let $G = \mathcal{S}_3$ be the symmetric group on three elements. For any fixed odd $n_2 \geq 1$ and $p \geq 0$, there is an isomorphism

$$H_p(\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3)}; \mathbb{Z}) \cong H_p(\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3+1)}; \mathbb{Z})$$

in the stable range $n > 35p + 31$.

We prove the following homological stability result for colored configuration spaces:

Theorem 4 (Theorem 2.24)
Let M be a connected orientable manifold of finite type. Then for any fixed $p \geq 0$, there is an isomorphism

$$H_p(\mathrm{Conf}_{n\cdot\underline{\xi}}(M);\mathbb{Q}) \cong H_p(\mathrm{Conf}_{(n+1)\cdot\underline{\xi}}(M);\mathbb{Q})$$

for $n \geq \frac{4p+\xi}{\max \underline{\xi}} - 1$ if $\dim M = 2$, and $n \geq \frac{2p+\xi}{\max \underline{\xi}} - 1$ if $\dim M > 2$.

In the study of homological stability, highly connected simplicial complexes are ubiquitous. In Chapter 4, we define *plants* and *colored plant complexes* in order to obtain Theorem 5, alongside further connectivity results which might become useful in future research.

Theorem 5 (Proposition 4.13, Lemma 4.14, Lemma 4.16, Lemma 4.18)
There exists an $(n-1)$-dimensional and at least $\left(\lfloor \frac{n}{2} \rfloor - 2\right)$-connected simplicial complex which admits a generally non-transitive action by the colored braid group $\mathrm{Br}_{n\cdot\underline{\xi}}$. The stabilizer of a q-simplex under this action is isomorphic to $\mathrm{Br}_{(n-q-1)\cdot\underline{\xi}}$.

1.2. Structure of the thesis

In Chapters 2 to 5, we work towards the proof of Theorem 1. After a general introduction to homological stability, we collect the necessary preliminaries about braid groups and configuration spaces and prove Theorem 4 in Chapter 2. We may then define the Hurwitz spaces we would like to consider in Chapter 3. In Chapter 4, we introduce and investigate *plant complexes*, with Theorem 5 as a result. This outcome enables us to define a spectral sequence converging to the homology of Hurwitz spaces in Chapter 5. The study of this spectral sequence culminates in the proof of Theorem 1. In Chapter 6, we apply the methods and results from the previous chapters in order to obtain further and more specific homological stability results, namely Theorem 2 and Theorem 3.

In two short appendices, we collect important definitions and results for simplicial complexes (Appendix A) and spectral sequences (Appendix B).

1.3. Notation and conventions

By the symbol $\mathbb{N}$, we denote the *positive* integers, while $\mathbb{N}_0$ is the set of *non-negative* integers. All rings are unital but not necessarily commutative. The letter I represents the real unit interval $[0, 1] \subset \mathbb{R}$.

Spaces and maps

- A *space* is a topological space. A *map* between spaces is continuous, unless otherwise mentioned.
- A *covering space* of a space need not be connected.
- By S^n and D^n we denote the real n-sphere and the real closed n-disk, respectively. We write D instead of D^2 for a closed two-dimensional disk.
- If G is a group, we denote by EG a contractible space with a free right G-action. The space $BG = EG/G$ is the *classifying space* of G, and both EG and BG exist in the category of CW complexes, well-defined up to homotopy. If G is discrete, BG is an Eilenberg-MacLane space of type $K(G, 1)$. For a connected CW complex X, there is a bijection between free homotopy classes of continuous maps $X \to BG$ and group homomorphisms $\pi_1(X) \to G$ up to conjugation in G. As references, cf. [May99, Ch. 16.5], [Hat02, Ch. 1.B], or [AM04, Ch. II.1].

Topological invariants Let X be a space, and $x \in X$.

- The p-th homotopy group of X with base point x is denoted by $\pi_p(X, x)$. If X is path-connected, we sometimes leave out the base point and write $\pi_p(X)$.
- We write $\operatorname{conn} X$ for the *connectivity* of X, i.e., $\operatorname{conn} X = -1$ if X is non-empty and disconnected, and $\operatorname{conn} X = \max\{i \geq 0 \mid \pi_j(X) = 1 \text{ for all } 0 \leq j \leq i\}$ otherwise.
- If X is a space and A a ring, $H_p(X; A)$ denotes the p-th singular homology module of X with coefficients in A. If G is a (discrete) group, $H_p(G; A)$ is the p-th group homology module of X, i.e., $H_p(G; A) := H_p(BG; A)$. We write $b_p(X) = \dim_{\mathbb{Q}} H_p(X; \mathbb{Q})$ for the p-th Betti number of X.
- The corresponding (singular/group) cohomology modules are obtained by passing from sub- to superscripts in the last item.
- Two spaces X and Y are of the same *homotopy type* if they are homotopy equivalent, and we write $X \simeq Y$.

Manifolds In this thesis, all manifolds M are real and smooth.

- M is *open* if its closure has non-empty boundary ∂M, and *closed* otherwise. A *compact manifold* is a manifold which is compact as a topological space. M is of *finite type* if its rational cohomology modules are finite-dimensional.
- A *surface* is a connected manifold of dimension two. A *Riemann surface* is an orientable surface together with a conformal structure.

Isotopies Let S be a compact surface. We have the following results, cf. also the discussion in [FM12, Ch. 1.4]:

- Every homeomorphism of S is isotopic to a diffeomorphism, cf. [Mun60] and [Whi61].
- If φ, φ' are orientation-preserving homeomorphisms of S which are homotopic, they are isotopic, cf. [Eps66].

Groups and group actions Let G be a group. Depending on the context, 1 denotes either the neutral element of a group G or the trivial group.

- We write F_n for the free group on n elements.
- $\mathcal{S}_n$ is the symmetric group on n elements, and $\mathcal{A}_n$ the corresponding alternating group. For permutations, we use the cycle notation; i.e., we write (132) for the permutation $\left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right) \in \mathcal{A}_n \subset \mathcal{S}_n$. Its neutral element is denoted by $\mathrm{id} \in \mathcal{S}_n$.
- $\mathrm{ord}(g)$ is the order of $g \in G$, while $|G|$ is the order of G.
- For $g, h \in G$, we write $g^h = h^{-1}gh$ for the conjugation operation. The inner automorphism $g \mapsto g^h$ is denoted by the symbol $()^h$.
- If a set X admits a left G-action, we denote by G_x the stabilizer of $x \in X$ and by $G \cdot x$ the orbit of x under the G-action. X^G denotes the G-invariants and X/G the set of G-orbits in X.

Graded modules Let R be a ring.

- The free left R-module generated by a set $\mathbf{c}$ is denoted by $R\langle \mathbf{c} \rangle$.
- For a left (right) R-module M and an element $U \in R$, we write $M[U]$ for the *U-torsion* in M, i.e., the kernel of the endomorphism $m \mapsto Um$ ($m \mapsto mU$).

- A *graded ring* $R = \bigoplus_{p\geq 0} R_p$ is a ring that decomposes into a direct sum of abelian groups R_p, such that $R_pR_q \subset R_{p+q}$ for all $p, q \geq 0$. The ideal $\bigoplus_{p>0} R_i$ is denoted by $R_{>0}$.

- A *left (right) graded R-module M* is a left (right) module over a graded ring $R = \bigoplus_{p\geq 0} R_p$ that decomposes as $M = \bigoplus_{q\geq 0} M_q$ into a direct sum of abelian groups, such that $R_pM_q \subset M_{p+q}$ ($M_qR_p \subset M_{p+q}$) for all $p, q \geq 0$.

- Let $M = \bigoplus_{p\geq 0} M_q$ be a graded R-module. An element $m \in M_q \setminus \{0\}$ is called *homogeneous of degree q*, and we write $\deg(m) = q$. The *degree* of a graded module M is $\deg(M) = \sup\{q \geq 0 \mid M_p \neq \{0\}\}$, where $\deg(M) = \infty$ is possible. A nontrivial linear map f between graded R-modules M and N has *degree p* if for all $q \geq 0$, we have $f(M_q) \subset N_{p+q}$.

- For a graded module M over R and $i \geq 0$, we write $M(i)$ for the *shift* of M by i, i.e., $M(i)_j = M_{i+j}$.

Partitions and tuples For $t \in \mathbb{N}$, let $\underline{\mu} = (\mu_1, \dots, \mu_t) \in \mathbb{N}^t$ be a t-tuple. The tuple is a partition of $\mu = |\underline{\mu}| = \sum_{i=1}^{t} \mu_i$, and we write $\underline{\mu} \vdash \mu$. We call μ the *length* of $\underline{\mu}$.

- $\max \underline{\mu}$ ($\min \underline{\mu}$) is the largest (smallest) entry of $\underline{\mu}$.

- For $n \in \mathbb{N}$, $n \cdot \underline{\mu}$ denotes the tuple $(n\mu_1, \dots, n\mu_t) \vdash n\mu$.

- $(\mu_1, \dots, \hat{\mu}_i, \dots, \mu_t)$ is the $(t-1)$-tuple obtained from $\underline{\mu}$ by removing μ_i.

Let G be a group, and $\underline{g} = (g_1, \dots, g_n) \in G^n$ for $n \in \mathbb{N}$.

- For $g \in G$, $(g^{(n)})$ is short for $\underbrace{(g, \dots, g)}_{n \text{ times}}$. Such a tuple is called *constant*.

- We denote by $\underline{g}^h$ the result of elementwise conjugation of $\underline{g}$ by $h \in G$.

- The *boundary* of $\underline{g}$ is the product $\partial \underline{g} = g_1 \cdot \ldots \cdot g_n$.

- If $\underline{h} = (h_1, \dots, h_m) \in G^m$, we write $(\underline{g}, \underline{h}) \in G^{n+m}$ for the $(n+m)$-tuple $(g_1, \dots, g_n, h_1, \dots, h_m)$.

Let now $\mathbf{c} \subset G^\xi$ and $\underline{g} \in \mathbf{c}^n \subset G^{n\xi}$.

- We write $(\underline{g})^{\leq j} \in \mathbf{c}^j$ for the tuple consisting of the first $j\xi$ entries of $\underline{g}$. The complementary $(n-j)\xi$-tuple is denoted by $(\underline{g})^{>j} \in \mathbf{c}^{n-j}$.

- $(\underline{g})_j \in \mathbf{c}$ is the j-th ξ-tuple in $\underline{g}$.

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2. Homological Stability

This chapter serves as an introduction to homological stability with a focus on configuration spaces and braid groups. To this end, in Section 2.1 we give an overview on the most important results needed for our investigation of Hurwitz spaces. In Section 2.2, after introducing the concept of representation stability, we prove a homological stability theorem for colored configuration spaces (Proposition 2.24). Finally, we provide a brief survey about homological stability for moduli spaces of Riemann surfaces in Section 2.3.

The concept Let $\{X_n \mid n \geq 0\}$ be a sequence of spaces or groups and $H_*(-)$ a suitable homology theory. We say that $\{X_n \mid n \geq 0\}$ satisfies *homological stability* with *stable range* $n \geq r(p)$ if for any $p \geq 0$, we have isomorphisms

$$H_p(X_n) \cong H_p(X_{n+1})$$

in homology whenever $n \geq r(p)$, where $r\colon \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ is a function in p. In many cases, r grows linearly or is constant.

Sometimes, there are *stabilizing maps* $s_n\colon X_n \to X_{n+1}$ for all $n \geq 0$ and isomorphisms

$$(s_n)_*\colon H_p(X_n) \xrightarrow{\sim} H_p(X_{n+1})$$

for $n \geq r(p)$. In other cases, there is only a fixed $u \in \mathbb{N}$ such that

$$H_p(X_n) \cong H_p(X_{n+u}).$$

in a stable range. We call this phenomenon *u-periodic homological stability.*

If $\{X_n \mid n \geq 0\}$ is homologically stable with stable range $n \geq r(p)$, the group $\bigoplus_{p\geq 0} H_p(X^p_\infty)$ is called the *stable homology* of the sequence. Here, for any fixed p, we choose $X^p_\infty = X_n$ for some $n \geq r(p)$.

From now on, we work exclusively with singular homology (for spaces) and group homology (for groups). Classical homological stability results include the theorems for symmetric groups $\mathcal{S}_n$, and general linear groups $\mathrm{GL}_n(R)$ for some ring R.

Theorem (NAKAOKA, [Nak60])
For any $p \geq 0$, there is an isomorphism $H_p(\mathcal{S}_n;\mathbb{Z}) \to H_p(\mathcal{S}_{n+1};\mathbb{Z})$ with stable range $n > 2p$. The stabilizing map is induced by embedding $\mathcal{S}_n$ into $\mathcal{S}_{n+1}$ as the stabilizer of the element $n+1$.

Theorem (MAAZEN, [Maa79]; VAN DER KALLEN, [vdK80])
Let R be a ring with stable rank e. For any $p \geq 0$, the map $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$ induces a map $H_p(\mathrm{GL}_n(R);\mathbb{Z}) \to H_p(\mathrm{GL}_{n+1}(R);\mathbb{Z})$ in group homology which is an isomorphism in the stable range $n \geq 2p + e$.

The basic proof strategy We consider the following typical situation, which is a special case of the setting in [HW10, Thm. 5.1]: Let $G_0 \hookrightarrow G_1 \hookrightarrow \dots$ be a sequence of discrete groups and X_n an Eilenberg-MacLane space of type $K(G_n, 1)$ for all $n \geq 0$. Hence, X_n has the homotopy type of the classifying space BG_n, and for all $n, p \geq 0$, we have isomorphisms $H_p(X_n) \cong H_p(G_n)$ between the singular homology modules of X_n and the group homology modules of G_n. In the following, we sketch a homological stability proof for the sequence $\{X_n \mid n \geq 0\}$, where we make convenient assumptions now and then and leave out some details.

Assume that for all $n \geq 0$ there is an ordered simplicial complex $\mathcal{O}^n$ with set of q-simplices $\mathcal{O}^n_q$ such that

(i) G_n acts transitively on $\mathcal{O}^n_q$,

(ii) the stabilizer subgroup $(G_n)_{\alpha_q}$ of a q-simplex $\alpha_q \in \mathcal{O}^n_q$ is conjugate to the subgroup $G_{n-q-1} \hookrightarrow G_n$, and

(iii) the geometric realization $|\mathcal{O}^n|$ is $s(n)$-connected, where $s\colon \mathbb{N} \to \mathbb{R}$ grows at least linearly in n.

The *arc complex* is an example for a suitable complex in the case where $G_n = \mathrm{Br}_n$ is the Artin braid group, cf. Remark 4.12.

Fix $n \in \mathbb{N}$. From the semi-simplicial structure on $\mathcal{O}^n$, the set $EG_n \times_{G_n} \mathcal{O}^n$ inherits a semi-simplicial structure with set of q-simplices $EG_n \times_{G_n} \mathcal{O}^n_q$. The associated spectral sequence (B.1)

$$E^1_{qp} = H_p\left(EG_n \times_{G_n} \mathcal{O}^n_q;\mathbb{Z}\right) \Longrightarrow H_{p+q}\left(EG_n \times_{G_n} |\mathcal{O}^n|;\mathbb{Z}\right)$$

converges to the homology of the realization of the total complex. By condition (iii), the target is is isomorphic to $H_{p+q}(BG_n;\mathbb{Z}) \cong H_{p+q}(X_n;\mathbb{Z})$ for $p + q \leq s(n)$. On the other hand, conditions (i) and (ii) yield an identification of $\mathcal{O}_q^n$ with G_n/G_{n-q-1}. As a consequence, the fact that EG_n is a model for EG_{n-q-1}, together with the orbit-stabilizer theorem, implies that we have

$$E^1_{qp} \cong H_p(BG_{n-q-1};\mathbb{Z}) \cong H_p(X_{n-q-1};\mathbb{Z}).$$

Thus, this spectral sequence calculates the homology of X_n from the homology of the X_k, for $0 \leq k < n$.

The differential $\mathrm{d}\colon E^1_{qp} \to E^1_{q-1,p}$ is induced by the alternating sum of the face maps

$$EG_n \times_{G_n} \mathcal{O}_q^n \to EG_n \times_{G_n} \mathcal{O}_{q-1}^n.$$

If we assume that all face maps induce the same map in homology (which we do for now), it follows that $\mathrm{d}\colon E^1_{qp} \to E^1_{q-1,p}$ is the zero map if q is odd, and induced by the face map if q is even. Inductively, assume that for any $p < P$, there are isomorphisms $H_p(X_n;\mathbb{Z}) \cong H_p(X_{n+1};\mathbb{Z})$ in a range, induced by the face map.

Now consider E^2_{0P}: Since $\mathrm{d} = 0$ for q odd, we have $E^2_{0P} = E^1_{0P} \cong H_P(X_{n-1};\mathbb{Z})$. The induction hypothesis ensures that $E^2_{q,P+1-q} = 0$ holds for $q > 1$. Thus, there are no differentials going into or out of E^2_{0P}, hence $E^2_{0P} = E^\infty_{0P}$. In addition, we have $E^2_{q,P-q} = 0$ for $q > 0$, so E^∞_{0P} is the only nonvanishing graded piece of $H_P(X_n)$. In particular, the edge map $E^1_{0P} = H_P(X_{n-1};\mathbb{Z}) \to H_P(X_n;\mathbb{Z})$ is an isomorphism in a suitable range, which implies the induction step (an argument why the edge map is equal to the face map can be found in the proof of Theorem 5.7).

In fact, homological stability for sequences of groups – or, equivalently, their classifying spaces – holds in an even more general setting: In particular, condition (i) may be replaced by the requirement that the group action on the semi-simplicial set is only transitive *on vertices*, cf. [HW10, Thm. 5.1].

Remark 2.1. Though this strategy is our inspiring guideline, we run into a couple of problems when examining homological stability for Hurwitz spaces: First, the Hurwitz spaces we consider (cf. Chapter 3) are usually disconnected, so they are no Eilenberg-MacLane spaces. This can be fixed by using the fact that they are finite covers of $K(G,1)$ spaces, where G is a *colored braid group* (cf. Section 2.2.2). Secondly, there is no semi-simplicial set at hand which admits a well-behaved colored braid group action. We solve this problem by the definition and investigation of the highly connected *colored plant complexes* in Chapter 4. Thirdly, the group action on these complexes is generally not transitive. This last point makes a more extensive homological analysis in Chapter 5 necessary.

2.1. Configurations and braids

Configuration spaces and braid groups are fundamental and closely related objects in (geometric) topology. In particular, they give rise to numerous homological stability theorems. There are various useful references; beyond the original papers, the reader may consult the monographs [Bir74], [Han89], [FH01], or [FM12] for further reading with varying focuses.

2.1.1. Foundations

Let M be a manifold with interior M° and D a closed two-dimensional disk.

Definition 2.2. Let $n \in \mathbb{N}$. The *ordered (or pure) configuration space of n points in M* is the open submanifold of $(M^\circ)^n$ defined by

$$\mathrm{PConf}_n(M) = \{(m_1, \ldots, m_n) \in (M^\circ)^n \mid m_i \neq m_j \text{ for } i \neq j\}.$$

The *(unordered) configuration space of n points in M* is defined as the quotient

$$\mathrm{Conf}_n(M) = \mathrm{PConf}_n(M)/\mathcal{S}_n,$$

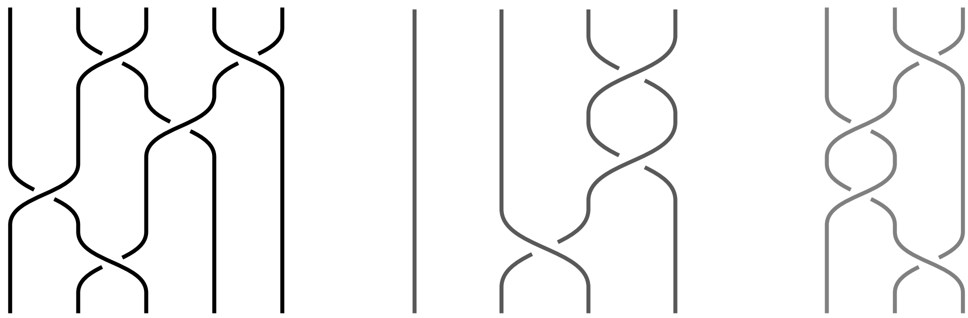
where the symmetric group $\mathcal{S}_n$ acts on $\mathrm{PConf}_n(M)$ by permuting the coordinates. For $M = D$, we simply write PConf_n and Conf_n.

Remark 2.3. The action of $\mathcal{S}_n$ on $\mathrm{PConf}_n(M)$ is free and properly discontinuous, so $\mathrm{Conf}_n(M)$ is a manifold topologized as a quotient of $\mathrm{PConf}_n(M) \subset (M^\circ)^n$. The quotient map $p_n\colon \mathrm{PConf}_n(M) \to \mathrm{Conf}_n(M)$ is an $\mathcal{S}_n$-Galois cover.

Definition 2.4. Let S be a surface and $n \in \mathbb{N}$. The group $\mathrm{Br}_n(S) = \pi_1(\mathrm{Conf}_n(S))$ is called the *surface braid group on n strands*. The *pure surface braid group on n strands* is defined by $\mathrm{PBr}_n(S) = \pi_1(\mathrm{PConf}_n(S))$.

Remark 2.5. For any fixed surface S, we obtain an inclusion $\mathrm{PBr}_n(S) \hookrightarrow \mathrm{Br}_n(S)$ from the cover $\mathrm{PConf}_n(S) \to \mathrm{Conf}_n(S)$ after choosing suitable base points.

In the following, we are mostly concerned with the case $M = D$. Up to homeomorphism, we may assume that D is of radius one, centered around the origin in the complex plane. We choose base points $\bar{C}_n \in \mathrm{PConf}_n$ and $C_n = p_n(\bar{C}_n) \in \mathrm{Conf}_n$ such that $\bar{C}_n$ has only real entries in the interior of D, and $C_{n-1} \subset C_n$ in such a way that the extra point in C_n lies to the right of C_{n-1}. We write $C_n = \{C^1, \ldots, C^n\} \subset D \cap \mathbb{R}$, where $C^i < C^j$ for $i < j$, and denote the punctured disk $D \setminus C_n$ by D_n.



$\sigma_2\sigma_4^{-1}\sigma_3\sigma_1\sigma_2^{-1} \in \mathrm{Br}_5$ $\quad$ $\sigma_3^2\sigma_2^{-1} \in \mathrm{Br}_4$ $\quad$ $\sigma_2^{-1}\sigma_1^2\sigma_2^{-1} \in \mathrm{PBr}_3$

Figure 2.1.: Visualization of Artin braids

Definition 2.6. Let $n \in \mathbb{N}$. A *strand* is a path in D with starting point and endpoint in C_n. An *(Artin) braid on n strands* is a loop in Conf_n with base point C_n. It is usually visualized as a projection to $(D \cap \mathbb{R}) \times I$ (Figure 2.1). The subset of $(D \cap \mathbb{R}) \times I$ defined by a braid is called *geometric braid*.

For a braid on n strands, its unique strand with starting point C^i is called the i-th strand. An Artin braid is called *pure* if all of its strands are loops in D.

The *(Artin) braid group on n strands* is the group Br_n of isotopy classes of Artin braids (isotopies relative $C_n \times \{0,1\}$) with the group operation "concatenating and rescaling", whereas the *pure (Artin) braid group* is the subgroup $\mathrm{PBr}_n \subset \mathrm{Br}_n$ of isotopy classes of pure Artin braids.

Remark 2.7. Definition 2.6 is a modification of the geometric definition of braids given in [Art25], where we highlight the connection to configuration spaces, which is implicitly present already in the work of Hurwitz, cf. [Hur91]. From our definition and the fact that homotopic loops are isotopic, the upcoming statement from Theorem 2.9 that Br_n is isomorphic to $\mathrm{Br}_n(D)$ is evident.

Definition 2.8. Let S be a surface with (possibly empty) boundary ∂S. The *mapping class group* $\mathrm{Map}(S)$ is the group of isotopy classes of orientation-preserving diffeomorphisms of S that fix ∂S pointwise:

$$\mathrm{Map}(S) = \pi_0 \left(\mathrm{Diff}^+ (S, \partial S) \right).$$

Note that the action of $\mathrm{Map}(S)$ may permute punctures of S.

As every orientation-preserving homeomorphism of S is homotopic to a diffeomorphism, $\mathrm{Map}(S) = \pi_0 (\mathrm{Homeo}^+ (S, \partial S))$. For a detailed primer on mapping class groups and related topics, including braids and configurations, cf. [FM12].

Theorem 2.9. *The following groups are isomorphic:*

(i) the Artin braid group Br_n*,*

(ii) the surface braid group $\mathrm{Br}_n(D) \cong \pi_1(\mathrm{Conf}_n, C_n)$*,*

(iii) the mapping class group $\mathrm{Map}(D_n)$*, and*

(iv) the group with generators $\sigma_1, \dots, \sigma_{n-1}$*, subject to the relations*

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, & 1 \le i \le n-2, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i, & |i-j| \ge 2. \end{aligned} \tag{2.1}$$

Proof. These results are well-known and can be found throughout the literature on braid groups. The presentation for Br_n was first developed in [Art25]. The other isomorphisms are straightforward, cf. [FM12, Ch. 9.1]. □

Consequently, we write $\mathrm{Br}_n = \mathrm{Br}_n(D)$, as already indicated in Remark 2.7. From now on, the phrase *braid group* usually indicates one of the isomorphic groups from Theorem 2.9, and in fact, different identifications prove useful in different situations.

In the following, we quickly describe the role of the generators σ_i in the three other incarnations of Br_n stated above, cf. Figure 2.2. Note that in the literature, the roles of σ_i and σ_i^{-1} are sometimes interchanged. We explain the unusual combination of choices in Remark 2.10.

(i) We identify σ_i with the geometric braid where the $(i+1)$-th strand passes in front of the i-th strand, the other strands being straight lines connecting $(C^j, 0)$ and $(C^j, 1)$ in $D \times I$, $j \notin \{i, i+1\}$, cf. Figure 2.2a.

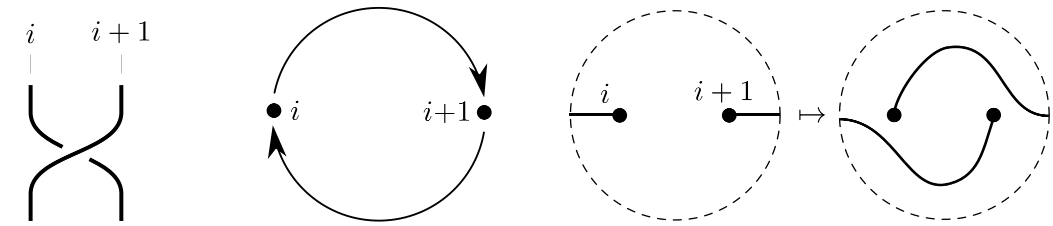
(ii) The Artin braid described in (i) determines a loop in Conf_n. The homotopy class of this loop defines an element of $\pi_1(\mathrm{Conf}_n, C_n)$, cf. Figure 2.2b.

(iii) Let $T_i \subset D$ be a small disk containing C^i and C^{i+1}. Then in $\mathrm{Map}(D_n)$, the generator σ_i corresponds to the isotopy class of a half counterclockwise twist of T_i which interchanges the two marked points, extended by the identity on $D \setminus T_i$, cf. Figure 2.2c.

We also see that we may identify

(i) the pure braid group PBr_n,

(ii) the pure surface braid group $\mathrm{PBr}_n(D) = \pi_1(\mathrm{PConf}_n, \bar{C}_n)$,



(a) geometric (b) loop in Conf_n (c) in $\mathrm{Map}(D_n)$

Figure 2.2.: Incarnations of the braid $\sigma_i \in \mathrm{Br}_n$

(iii) the subgroup of elements of $\mathrm{Map}(D_n)$ which fix C_n pointwise, and

(iv) the kernel of the surjection $\mathrm{Br}_n \to \mathcal{S}_n$ induced by mapping σ_i to the transposition $(i, i+1)$ (this is well-defined since the braid relations (2.1) are satisfied by transpositions in $\mathcal{S}_n$).

A finite presentation for PBr_n was first given in [Art47]. We obtain a short exact sequence

$$1 \to \mathrm{PBr}_n \to \mathrm{Br}_n \to \mathcal{S}_n \to 1. \tag{2.2}$$

We call $\mathrm{Br}_n \to \mathcal{S}_n$ the *permutation map*. In Section 2.2.2, we consider subgroups of braid groups which generalize this construction of the pure braid group.

Remark 2.10. The choices for the identifications of the generators do not seem to fit at first glance: As a loop in Conf_n, the braid σ_i interchanges C_i and C_{i+1} by rotating them in *clockwise order*. In contrast, the mapping class $\sigma_i \in \mathrm{Map}(D_n)$ denotes a half *counterclockwise* twist which interchanges C_i and C_{i+1}.

This ostensible contradiction is motivated by the usual notation for the groups: On the one hand, loops in fundamental groups are usually concatenated from left to right, which is also reflected in the fact that monodromy actions are most often written as a right action. On the other hand, diffeomorphisms mostly act from the left.

Any diffeomorphism $\sigma \in \mathrm{Diff}^+(D_n, \partial D_n)$ defines an element of $\mathrm{Diff}^+(D, \partial D)$. This group is connected (cf. [FM12, Lemma 2.1]), so there is an isotopy in $\mathrm{Diff}^+(D, \partial D)$ from the identity to σ. The trace of C_n under this isotopy defines a loop in Conf_n. If we understand σ as an element of $\mathrm{Br}_n \cong \mathrm{Map}(D_n)$, the homotopy class of the corresponding loop in Conf_n is now given by $\sigma^{-1} \in \mathrm{Br}_n \cong \pi_1(\mathrm{Conf}_n, C_n)$.

Furthermore, we may construct a left action $\sigma \cdot x \coloneqq x * \sigma^{-1}$ from any right action $x \mapsto x * \sigma$. Hence in our construction, the left action of $\mathrm{Map}(D_n)$ on D_n is compatible with the *left* monodromy action of $\pi_1(\mathrm{Conf}_n, C_n)$ on any cover of D_n, using the correspondence from the last paragraph.

Note that also the permutation map $\mathrm{Br}_n \to \mathcal{S}_n$ depends on our identification of the braid group, cf. Remark 2.10: If we interpret the braid group as a fundamental group, we multiply cycles from left to right. Accordingly, if braids are understood as mapping classes, cycles are concatenated from right to left.

In fact, both described maps $\mathrm{Br}_n \to \mathcal{S}_n$ are homomorphic. This is caused by the fact that the braid relations are symmetric in the generators σ_i, so $\sigma_i \mapsto \sigma_i^{-1}$ can be extended to an automorphism of Br_n.

The braid group Br_n is not only isomorphic to the fundamental group of Conf_n. In fact, Conf_n classifies the braid group, so the singular homology of Conf_n is isomorphic to the group homology of Br_n:

Theorem 2.11 (Fadell–Neuwirth, [FN62])
The space Conf_n *is aspherical, i.e.,* $\pi_i(\mathrm{Conf}_n, C_n) = 1$ *for* $i \geq 2$. *In other words,* Conf_n *has the homotopy type of the classifying space* $B\mathrm{Br}_n$.

Remark 2.12. The Fadell-Neuwirth theorem, together with the long exact homotopy sequence for the covering space map p_n with fiber $\mathcal{S}_n$, yields another derivation of the sequence (2.2). From the long exact sequence, we obtain that also PConf_n is aspherical.

2.1.2. Homological stability results

There are several results about homological stability for configuration spaces in various settings, and there is little chance to give a satisfactory overview. For this reason, we stick to the results which fit into the common theme of this thesis; this in particular excludes results which deal exclusively with manifolds of dimension greater than two, or non-orientable manifolds.

At first, we look for a potential stabilization map for $M = D$. It is clear that there exists an inclusion $\tilde{\alpha}\colon \mathrm{Br}_n \to \mathrm{Br}_{n+1}$ defined by attaching the strand given by the trivial loop at C^{n+1}. As Conf_n classifies Br_n, the map $\tilde{\alpha}$ induces a unique homotopy class of based maps $\alpha\colon (\mathrm{Conf}_n, C_n) \to (\mathrm{Conf}_{n+1}, C_{n+1)}$ which corresponds to a unique map α_* in homology.

Geometrically, α does the following: Given a disk D with n marked points, consider an annulus with one marked point and glue one of its boundaries to ∂D. Then, shrink the larger disk with $n+1$ marked points to the original size of D. Visibly, this procedure works in general for open manifolds. In fact, we have the following result:

Theorem 2.13 (ARNOL'D, [Arn70])
For any $p \geq 0$, $\alpha_*\colon H^p(\mathrm{Conf}_n;\mathbb{Z}) \cong H^p(\mathrm{Conf}_{n+1};\mathbb{Z})$ *for* $n \geq 2p-2$. *Furthermore,* $H^0(\mathrm{Conf}_n;\mathbb{Q}) \cong H^1(\mathrm{Conf}_n;\mathbb{Q}) \cong \mathbb{Q}$ *are the only nontrivial rational cohomology groups of* Conf_n.

For general open manifolds M, homological stability for configuration spaces was proved in the 70s. Since then, the result by McDuff and Segal has been generalized by Randal-Williams (with the same stable range, cf. [RW13]) to the case of configurations of open manifolds with *labels*.

Theorem (MCDUFF, [McD75]; SEGAL, [Seg79])
Let M *be an open manifold,* $\dim M \geq 2$. *Then, for any fixed* $p \geq 0$, *the stabilization map* $\alpha_*\colon H_p(\mathrm{Conf}_n(M);\mathbb{Z}) \cong H_p(\mathrm{Conf}_{n+1}(M);\mathbb{Z})$ *is an isomorphism for* $n \geq 2p$.

For closed manifolds, we cannot define the stabilization map α in an analogous way. For ordered configurations, the situation is somewhat different: At least, there is a map $\mathrm{PConf}_{n+1}(M) \to \mathrm{PConf}_n(M)$ that forgets the last point of a configuration. Church shows in [Chu12] that this map induces *representation stability* (Theorem 2.17), which implies the theorem about the rational homology of configuration spaces of arbitrary manifolds quoted below.

Theorem 2.14 (CHURCH, [Chu12])
Let M *be a connected orientable manifold of finite type of dimension* ≥ 2. *For any* $p \geq 0$, *there is an isomorphism* $H_p(\mathrm{Conf}_n(M);\mathbb{Q}) \cong H_p(\mathrm{Conf}_{n+1}(M);\mathbb{Q})$ *in the stable range* $n \geq p+1$.

2.2. Colored configuration spaces

2.2.1. Representation stability

The relatively new concept of *representation stability* was introduced by Church and Farb in [CF13]. The idea is to relate questions about homological stability to representation-theoretic concepts. More recently, this theory has been placed in the larger framework of the theory of FI-modules, which is developed in [CEF15].

The motivating example for representation stability is the following: As already seen in Theorem 2.13, we know from Arnol'd that the sequence $\{\mathrm{Br}_n \mid n \geq 0\}$ of (Artin) braid groups satisfies integral homological stability. However, this is not true for pure braid groups, whose rational cohomology rings are explicitly described in [Arn69]. Homological stability already fails for the sequence of first homology groups: $H_1(\mathrm{PBr}_n; \mathbb{Q}) \cong \mathbb{Q}^{\binom{n}{2}}$.

As PConf_n classifies PBr_n, their cohomology modules coincide. Now, the symmetric group $\mathcal{S}_n$ acts freely on PConf_n, giving $H^p(\mathrm{PConf}_n; \mathbb{Q}) = H^p(\mathrm{PBr}_n; \mathbb{Q})$ the structure of an $\mathcal{S}_n$-representation for any $i \geq 0$. As a representation of a finite group, we may thus consider the decomposition of $H^p(\mathrm{PConf}_n; \mathbb{Q})$ into irreducible $\mathcal{S}_n$-representations.

The irreducible representations of $\mathcal{S}_n$ are in bijection to the partitions $\underline{\lambda} \vdash n$ (for details about the representation theory of symmetric groups, cf. [CSST10]). We write $V_{\underline{\lambda}}$ for the irreducible representation over $\mathbb{Q}$ corresponding to $\underline{\lambda} \vdash n$; in particular, $V_{(n)} \cong \mathbb{Q}$ is the trivial irreducible $\mathcal{S}_n$-representation. Furthermore, each partition $\underline{\mu} = (\mu_1, \ldots, \mu_t)$ (the μ_i chosen in non-increasing order) defines a partition of n given by $\left(n - |\underline{\mu}|, \mu_1, \ldots, \mu_t\right) \vdash n$ as long as $n - |\underline{\mu}| \geq \mu_1$, where we write $|\underline{\mu}| = \sum_{i=1}^t \mu_i$. We denote the corresponding irreducible $\mathcal{S}_n$-representation by $V(\underline{\mu})_n$. Now for all $n \geq 4$, we have

$$H^1(\mathrm{PConf}_n; \mathbb{Q}) = V(0)_n \oplus V(1)_n \oplus V(2)_n,$$

and explicit calculations suggest that similar stability statements might hold for higher cohomologies, cf. [CF13, Sect. 1]. This leads to the following definition from [CF13].

Definition 2.15. Let $\{V_n \mid n \geq 0\}$ be a sequence of $\mathcal{S}_n$-representations.

(i) $\{V_n \mid n \geq 0\}$ is called *consistent* if there are linear maps $\phi_n \colon V_n \to V_{n+1}$ such that $\sigma \cdot \phi_n(v) = \phi_n(\sigma \cdot v)$ for all $v \in V_n$ and $\sigma \in \mathcal{S}_n$, where σ acts on V_{n+1} via the standard inclusion $\mathcal{S}_n \hookrightarrow \mathcal{S}_{n+1}$.

(ii) Let $\{V_n \mid n \geq 0\}$ be a consistent sequence of $\mathcal{S}_n$-representations. We say that the sequence is *uniformly representation stable with stable range $n \geq N$* if for all $n \geq N$, the three following conditions hold:

  (I) The maps ϕ_n are injective,

  (II) V_{n+1} is spanned by $\phi_n(V_n)$ as an $\mathcal{S}_{n+1}$-module, and

  (III) for any partition $\underline{\lambda}$, the multiplicity $c_{\underline{\lambda}}(V_n)$ of the irreducible representation $V(\underline{\lambda})_n$ in V_n is independent of n.

Indeed, there is the following result:

Theorem 2.16 (Church–Farb, [CF13])
For any fixed $p \geq 0$, the sequence $\{H^p(\mathrm{PConf}_n; \mathbb{Q}) \mid n \geq 0\}$ of $\mathcal{S}_n$-representations is uniformly representation stable with stable range $n \geq 4p$.

This gives a different proof of rational homological stabilization for the braid groups: Using the transfer homomorphism (cf. [Hat02, Prop. 3G.1]) associated to the covering space map $\mathrm{PConf}_n \to \mathrm{Conf}_n$ induced by the $\mathcal{S}_n$-action on PConf_n, we obtain

$$H^p(\mathrm{Conf}_n; \mathbb{Q}) \cong H^p(\mathrm{PConf}_n; \mathbb{Q})^{\mathcal{S}_n} \tag{2.3}$$

for all $p \geq 0$. The $\mathcal{S}_n$-invariants are exactly the copies of the one-dimensional trivial representation $V_{(n)} = V(0)_n$ in $H^p(\mathrm{PConf}_n; \mathbb{Q})$. By Theorem 2.16, the multiplicity of $V(0)_n \cong \mathbb{Q}$ in $H^p(\mathrm{PConf}_n; \mathbb{Q})$ is independent of n once $n \geq 4p$; in other words, $\dim_{\mathbb{Q}} H^p(\mathrm{Conf}_n; \mathbb{Q})$ is constant in this range. As a consequence of the universal coefficient theorem for cohomology (cf. [Hat02, Prop. 3.2]), we have

$$\dim_{\mathbb{Q}} H^p(\mathrm{Conf}_n; \mathbb{Q}) = \dim_{\mathbb{Q}} H_p(\mathrm{Conf}_n; \mathbb{Q}),$$

so we eventually obtain rational homological stability for the sequence $\{\mathrm{Conf}_n \mid n \geq 0\}$ with stable range $n \geq 4p$. This is of course far from the optimal result which states rational homological stability for $n \geq 0$ independently of p (Theorem 2.13).

We conclude this introductory section about representation stability with Church's result about the rational cohomology of ordered configuration spaces. By the above reasoning, we see that this theorem serves as the main ingredient for the proof of Theorem 2.14. The better stable range in the result for unordered configurations demands some extra work which is carried out in the article.

Theorem 2.17 (Church, [Chu12])
Let M be a connected orientable manifold of finite type, $\dim M \geq 2$. Then, for any fixed $p \geq 0$, the sequence $\{H^p(\mathrm{PConf}_n(M); \mathbb{Q}) \mid n \geq 0\}$ is uniformly representation stable with stable range $n \geq 4p$ if $\dim M = 2$, and $n \geq 2p$ if $\dim M > 2$.

2.2.2. Colored configurations

We now turn to *colored* configuration spaces, which are finite covers of unordered configuration spaces. If $\underline{\xi} = (\xi_1, \ldots, \xi_t) \in \mathbb{N}^t$ is a t-tuple, we write $\xi = \sum_{i=1}^t \xi_i$.

Definition 2.18. Let M be a manifold, and $\underline{\xi} = (\xi_1, \ldots, \xi_t) \in \mathbb{N}^t$. Let furthermore $\mathcal{S}_{\underline{\xi}}$ be the subgroup of $\mathcal{S}_\xi$ defined by $\mathcal{S}_{\underline{\xi}} = \mathcal{S}_{\xi_1} \times \ldots \times \mathcal{S}_{\xi_t}$, where $\mathcal{S}_{\xi_i}$ denotes the symmetric group on the elements $\left\{\left(\sum_{k=1}^{i-1} \xi_k\right) + 1, \left(\sum_{k=1}^{i-1} \xi_k\right) + 2, \ldots, \sum_{k=1}^{i} \xi_k\right\}$ for $i = 1, \ldots, t$. The *colored configuration space of ξ points in M with coloring $\underline{\xi}$* is defined as the quotient

$$\mathrm{Conf}_{\underline{\xi}}(M) = \mathrm{PConf}_\xi(M)/\mathcal{S}_{\underline{\xi}}.$$

For an integer $n \geq \xi$, we write $\mathrm{Conf}_{n,\underline{\xi}}(M) = \mathrm{PConf}_n(M)/\mathcal{S}_{(n-\xi,\xi_1,\ldots,\xi_t)}$. For $M = D$, we simply use the symbols $\mathrm{Conf}_{\underline{\xi}}$ and $\mathrm{Conf}_{n,\underline{\xi}}$.

Remark 2.19. Since the action of $\mathcal{S}_\xi$ on PConf_ξ is free and properly discontinuous, the same holds for any of its subgroups. We obtain a sequence of covers

$$\mathrm{PConf}_\xi(M) \to \mathrm{Conf}_{\underline{\xi}}(M) \to \mathrm{Conf}_\xi(M). \tag{2.4}$$

Clearly, $\mathrm{Conf}_{\underline{\xi}}(M)$ is the ordered configuration space $\mathrm{PConf}_\xi(M)$ for $\underline{\xi} = (1,\ldots,1)$ and the unordered configuration space $\mathrm{Conf}_\xi(M)$ for $\underline{\xi} = (\xi)$.

Let $M = D$. For any tuple $\underline{\xi}$, (2.4) corresponds to a sequence $\mathrm{PBr}_\xi \hookrightarrow \mathrm{Br}_{\underline{\xi}} \hookrightarrow \mathrm{Br}_\xi$ of group inclusions, where $\mathrm{Br}_{\underline{\xi}} \cong \pi_1(\mathrm{Conf}_{\underline{\xi}})$ is the preimage of $\mathcal{S}_{\underline{\xi}} \subset \mathcal{S}_\xi$ under the permutation map $p\colon \mathrm{Br}_\xi \to \mathcal{S}_\xi$. Evidently, $\mathrm{Conf}_{\underline{\xi}}$ is of type $B\mathrm{Br}_{\underline{\xi}}$, which follows from the fact that PConf_ξ is of type $B\mathrm{PBr}_\xi$ (cf. Remark 2.12) and the long exact homotopy sequence for $\mathrm{PConf}_\xi \to \mathrm{Conf}_{\underline{\xi}}$.

Definition 2.20. The group $\mathrm{Br}_{\underline{\xi}} \subset \mathrm{Br}_\xi$ is called the *colored braid group on ξ strands with coloring $\underline{\xi}$.*

Remark 2.21. Presentations for the groups $\mathrm{Br}_{\underline{\xi}}$ are established in [Man97] and, together with presentations for more general subgroups of Br_ξ, in [Lön10].

There are at least two homological stability results for colored configuration spaces, or (equivalently) colored braid groups: Church's result (Theorem 2.22) is deduced by representation stability methods, while Tran's proof of Theorem 2.23 uses more classical approaches to the subject such as the analysis of fiber bundles and their associated spectral sequences.

Theorem 2.22 (Church, [Chu12])
Let M be a connected orientable manifold of finite type, $\dim M \geq 2$. Then for any fixed $p \geq 0$, there is an isomorphism $H_p(\mathrm{Conf}_{n,\underline{\xi}}(M);\mathbb{Q}) \cong H_p(\mathrm{Conf}_{n+1,\underline{\xi}}(M);\mathbb{Q})$ for $n \geq \max\{4p, 2\xi\}$ if $\dim M = 2$, and $n \geq \max\{2p, 2\xi\}$ if $\dim M > 2$.

For $i = 1,\ldots,t$, let $e_i \in \mathbb{Z}^t$ be the i-th standard unit vector.

Theorem 2.23 (Tran, [Tra13])
Let M be a connected open manifold of $\dim M \geq 2$ whose closure is compact. Then for any fixed $p \geq 0$ and any $i = 1,\ldots,t$, $H_p(\mathrm{Conf}_{\underline{\xi}}(M);\mathbb{Z}) \cong H_p(\mathrm{Conf}_{\underline{\xi}+e_i}(M);\mathbb{Z})$ in the stable range $\xi_i \geq 2p$.

For later applications, we need a stability result for *diagonal* directions: From now on, let $\underline{\xi}$ be a fixed t-tuple of positive integers and $n \cdot \underline{\xi} = (n\xi_1, \ldots, n\xi_t)$. We consider the colored configuration spaces $\mathrm{Conf}_{n\cdot\underline{\xi}}(M)$ and want to stabilize in the n-direction. Indeed, a homological stability result for open manifolds follows directly from subsequent application of Theorem 2.23: We have

$$H_p(\mathrm{Conf}_{n\cdot\underline{\xi}}(M);\mathbb{Z}) \cong H_p(\mathrm{Conf}_{(n+1)\cdot\underline{\xi}}(M);\mathbb{Z}) \quad \text{for } n \geq \frac{2p}{\min \underline{\xi}} \tag{2.5}$$

Arguments from representation theory allow us to prove the following proposition for more general manifolds. In order to allow closed manifolds, we need to approve rational coefficients. We note that for open manifolds, our rational stable range improves on (2.5) in many cases (if $\dim M = 2$, roughly for $\max \underline{\xi} > 2 \min \underline{\xi}$).

Proposition 2.24. *Let M be a connected orientable manifold of finite type. Then for any fixed $p \geq 0$, there is an isomorphism $H_p(\mathrm{Conf}_{n\cdot\underline{\xi}}(M);\mathbb{Q}) \cong H_p(\mathrm{Conf}_{(n+1)\cdot\underline{\xi}}(M);\mathbb{Q})$ for $n \geq \frac{4p+\xi}{\max \underline{\xi}} - 1$ if $\dim M = 2$, and $n \geq \frac{2p+\xi}{\max \underline{\xi}} - 1$ if $\dim M > 2$.*

Proof. The present proof is inspired by the proofs of [Chu12, Thm. 5] (cf. Theorem 2.22) and [Chu12, Lemma 5.2]. We treat the case $\dim M = 2$, with the higher dimensional case following analogously, using the different stable range from Theorem 2.17. We may assume that we have $\xi_1 = \max \underline{\xi}$.

By a transfer argument as in (2.3), $H^p(\mathrm{Conf}_{n\cdot\underline{\xi}}(M);\mathbb{Q}) \cong H^p(\mathrm{PConf}_{n\xi}(M);\mathbb{Q})^{\mathcal{S}_{n\cdot\underline{\xi}}}$. In other words, the p-th rational cohomology of $\mathrm{Conf}_{n\cdot\underline{\xi}}(M)$ is isomorphic to the direct sum of the trivial $\mathcal{S}_{n\cdot\underline{\xi}}$-representations in $H^p(\mathrm{PConf}_{n\xi}(M);\mathbb{Q})$. By Theorem 2.17, the sequence $\{H^p(\mathrm{PConf}_{n\xi}(M);\mathbb{Q}) \mid n \geq 0\}$ is uniformly representation stable for any $p \geq 0$, with stable range $n\xi \geq 4p$. In particular, it follows for any partition $\underline{\lambda}$ with $|\underline{\lambda}| \geq 4p$ that the multiplicity of $V(\underline{\lambda})_{n\xi}$ in $H^p(\mathrm{PConf}_{n\xi}(M);\mathbb{Q})$ is zero for all n (recall that $V(\underline{\lambda})_{n\xi}$ is defined only for $n\xi - |\underline{\lambda}| \geq \lambda_1$).

It remains to show that for an increasing n, the multiplicity of the trivial $\mathcal{S}_{n\cdot\underline{\xi}}$-representation in the irreducible $\mathcal{S}_{n\xi}$-representation $V(\underline{\lambda})_{n\xi}$ becomes eventually constant for any partition $\underline{\lambda} = (\lambda_1, \ldots, \lambda_l)$ with $|\underline{\lambda}| < 4p$. Then, the equidimensionality of homology and cohomology groups implies the result.

Frobenius reciprocity implies that the multiplicity of the $\mathcal{S}_{n\cdot\underline{\xi}}$-representation $V(0)_{n\cdot\underline{\xi}}$ in $\mathrm{Res}^{\mathcal{S}_{n\xi}}_{\mathcal{S}_{n\cdot\underline{\xi}}} V(\underline{\lambda})_{n\xi}$, the restriction to $\mathcal{S}_{n\cdot\underline{\xi}}$ of the $\mathcal{S}_{n\xi}$-representation $V(\underline{\lambda})_{n\xi}$, is equal to the multiplicity of $V(\underline{\lambda})_{n\xi}$ in the induced $\mathcal{S}_{n\xi}$-representation $\mathrm{Ind}^{\mathcal{S}_{n\xi}}_{\mathcal{S}_{n\cdot\underline{\xi}}} V(0)_{n\cdot\underline{\xi}}$. We prove stability for the latter multiplicity. For the definition of the induced representation and a reference for Frobenius reciprocity, cf. [CSST10, Ch. I.6] or [Chu12, Sect. 2.2].

Any $\mathcal{S}_a$-representation V induces an $(\mathcal{S}_a \times \mathcal{S}_b)$-representation by letting $\mathcal{S}_b$ act trivially. We denote this representation by $V \boxtimes \mathbb{Q}$. Now, for the decomposition of induced representations, we have Pieri's formula (cf. [CSST10, Cor. 3.5.14]):

$$\operatorname{Ind}_{\mathcal{S}_a \times \mathcal{S}_b}^{\mathcal{S}_{a+b}} V_{\underline{\kappa}} \boxtimes \mathbb{Q} = \bigoplus_{\substack{\underline{\mu} \vdash n \\ \underline{\kappa} \rightsquigarrow \underline{\mu}}} V_{\underline{\mu}}$$

Here, we write $\underline{\kappa} \rightsquigarrow \underline{\mu}$ if the Young diagram for $\underline{\mu}$ is obtained from the Young diagram for $\underline{\kappa}$ by adding b boxes, adding no two boxes in the same column.

Now, repeated application of Pieri's formula enables us to calculate the decomposition of $\operatorname{Ind}_{\mathcal{S}_{n\cdot\underline{\xi}}}^{\mathcal{S}_{n\xi}} V(0)_{n\cdot\underline{\xi}}$ into irreducible $\mathcal{S}_{n\xi}$-representations. This works since by definition, we have $V(0)_{n\cdot\underline{\xi}} = \underbrace{\mathbb{Q} \boxtimes \ldots \boxtimes \mathbb{Q}}_{t \text{ times}}$ and

$$\operatorname{Ind}_{\mathcal{S}_a \times \mathcal{S}_b \times \mathcal{S}_c}^{\mathcal{S}_{a+b+c}} \mathbb{Q} \boxtimes \mathbb{Q} \boxtimes \mathbb{Q} = \operatorname{Ind}_{\mathcal{S}_{a+b} \times \mathcal{S}_c}^{\mathcal{S}_{a+b+c}} \left(\left(\operatorname{Ind}_{\mathcal{S}_a \times \mathcal{S}_b}^{\mathcal{S}_{a+b}} \mathbb{Q} \boxtimes \mathbb{Q} \right) \boxtimes \mathbb{Q} \right).$$

Therefore, the multiplicity of $V(\underline{\lambda})_{n\xi}$ in $\operatorname{Ind}_{\mathcal{S}_{n\cdot\underline{\xi}}}^{\mathcal{S}_{n\xi}} V(0)_{n\cdot\underline{\xi}}$ is given by the number of sequences $(\underline{\nu}^1, \ldots, \underline{\nu}^t)$ of partitions such that

- $\underline{\nu}^1 = (n\xi_1)$,
- $|\underline{\nu}^{i+1}| = |\underline{\nu}^i| + n\xi_{i+1}$ and $\underline{\nu}^i \rightsquigarrow \underline{\nu}^{i+1}$, and
- $\underline{\nu}^t = (n\xi - |\underline{\lambda}|, \lambda_1, \ldots, \lambda_l)$.

We call such a sequence *valid for* $V(\underline{\lambda})_{n\xi}$.

Let $(\underline{\nu}^1, \ldots, \underline{\nu}^k)$ be valid for $V(\underline{\lambda})_{n\xi}$. Now, consider the sequence of partitions given by $(\underline{\eta}^1, \ldots, \underline{\eta}^k)$, where $\underline{\eta}^i$ is constructed from $\underline{\nu}^i$ by adding $\sum_{j=1}^i \xi_j$ to the first (and largest) entry of $\underline{\nu}^i$. Visibly, this new sequence is valid for $V(\underline{\lambda})_{(n+1)\cdot\xi}$.

On the other hand, we can invert this process: Let $(\underline{\eta}^1, \ldots, \underline{\eta}^k)$ be a valid sequence for $V(\underline{\lambda})_{(n+1)\cdot\xi}$. For $n \geq \frac{4p+\xi}{\max \underline{\xi}} - 1$ (and this implies $n \geq \frac{4p}{\xi}$), we have

$$\eta_1^i - \sum_{j=1}^i \xi_j \geq \eta_1^1 - \xi = (n+1) \cdot \xi_1 - \xi \geq 4p > |\underline{\lambda}| \geq \lambda_1 \geq \eta_2^i,$$

where η_j^i is the j-th entry of $\underline{\eta}^i$. That is, lowering the first entry of the i-th term of a valid sequence for $V(\underline{\lambda})_{(n+1)\cdot\xi}$ by $\sum_{j=1}^i \xi_j$ produces a valid sequence for $V(\underline{\lambda})_{n\xi}$. In other words, there is a bijection between valid sequences for $V(\underline{\lambda})_{n\xi}$ and for $V(\underline{\lambda})_{(n+1)\cdot\xi}$, so the multiplicities in question are constant for $n \geq \frac{4p+\xi}{\max \underline{\xi}} - 1$. $\square$

2.3. Moduli of Riemann surfaces

In Sections 2.1 and 2.2, we dealt with parametrizing spaces for finite subsets of surfaces. If we fix the genus and the number of boundary components, there is only one surface up to homeomorphism. Hence, in order to obtain an interesting moduli theory for surfaces, one has to use more structure. The question in which way complex structures may vary leads to the theory of moduli spaces of compact Riemann surfaces (or, equivalently, of complex algebraic curves). In here, we shortly describe the complex analytic construction of these moduli spaces via Teichmüller spaces – for more details and different constructions, cf. [HM98] or [ACG11]. Another thorough presentation of our construction can be found in [FM12, Ch. 12].

In the following, let $S_{g,r}$ be a surface of genus $g \geq 2$ with $r \geq 0$ boundary components. We define the *moduli space of Riemann surfaces* $\mathcal{M}_{g,r}$ as the space of conformal structures on $S_{g,r}$ up to diffeomorphism, where $\mathrm{Diff}^+(S_{g,r}, \partial S_{g,r})$ acts on conformal structures by pullback. The space of conformal structures on $S_{g,r}$ up to isotopy is called *Teichmüller space*, denoted by $\mathcal{T}_{g,r}$. In order to obtain $\mathcal{M}_{g,r}$, it remains to quotient out by the action of $\pi_0(\mathrm{Diff}^+(S_{g,r}, \partial S_{g,r}))$, which is exactly $\mathrm{Map}(S_{g,r})$. Hence,

$$\mathcal{M}_{g,r} = \mathcal{T}_{g,r} / \mathrm{Map}(S_{g,r})$$

may be constructed as a quotient of Teichmüller space. Now, $\mathcal{T}_{g,r}$ is homeomorphic to $\mathbb{R}^{6g-6+2r}$, proved in [Ber60] and [FK65]. Ideally, $\mathrm{Map}(S_{g,r})$ would act freely and properly discontinuously on $\mathcal{T}_{g,r}$. Indeed, the action is properly discontinuous, cf. [FK65]. The stabilizer of an isotopy class of Riemannian metrics is exactly its group of automorphisms (up to isotopy), which is finite for $g \geq 2$ by Hurwitz' automorphisms theorem (cf. [Hur92]) and trivial for $r > 0$. This gives $\mathcal{M}_{g,r}$ an orbifold structure. From the finiteness of the stabilizers we obtain that the rational homology of $\mathcal{M}_{g,r}$ is isomorphic to the rational equivariant homology of $\mathcal{T}_{g,r}$ (cf. [Bri98, Sect. 1]). As Teichmüller space is contractible, $E\mathrm{Map}(S_{g,r}) \times \mathcal{T}_{g,r}$ is a model for $E\mathrm{Map}(S_{g,r})$, so

$$H_p(\mathcal{M}_{g,r}; \mathbb{Q}) \cong H_p(\mathrm{Map}(S_{g,r}); \mathbb{Q}).$$

We may now start wondering about homological stabilization for mapping class groups of surfaces. The corresponding theorem was proved by Harer with a stable range of $g \geq 3p$. Subsequently, various authors improved on this, cf. [Iva87], [Iva89], [Iva93], [Bol12], and most recently [RW16]. A survey of the theorem's background and a detailed proof for the most recent stable range can be found in [Wah13].

Theorem 2.25 (Harer, [Har85])
For any $p \geq 0$, there is an isomorphism $H_p(\mathrm{Map}(S_{g,r}); \mathbb{Z}) \cong H_p(\mathrm{Map}(S_{g+1,r}); \mathbb{Z})$ with stable range $g \geq \frac{3}{2}p + 1$.

From the above considerations, we obtain from Harer's theorem that for any fixed $r \geq 0$, the sequence $\{\mathcal{M}_{g,r} \mid g \geq 0\}$ satisfies rational homological stability with stable range $g \geq \frac{3}{2}p + 1$.

Let us now stick to the case of closed surfaces. Having established the above homological stability result, one might wonder what the stable rational (co-)homology looks like. In [Mum83], Mumford defined classes $\kappa_i \in H^{2i}(\mathrm{Map}(S_g); \mathbb{Q})$, now known as *Mumford-Morita-Miller classes*, and conjectured the statement of Theorem 2.26, which became known as the *Mumford conjecture*. It is now a theorem by Madsen and Weiss, the proof of which has been simplified in the later article [GTMW09].

Theorem 2.26 (Madsen–Weiss, [MW07])
For $g \geq 2$, the cohomology ring $H^*(\mathrm{Map}(S_g); \mathbb{Q})$ is isomorphic to a polynomial algebra $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ in the Mumford-Morita-Miller classes in a range of dimensions growing with g (the stable range coming from the Harer theorem).

3. Hurwitz Spaces

In this chapter, we introduce the central objects of our interest: Hurwitz spaces are parametrizing spaces for branched covers of a fixed surface and closely linked to both configuration spaces and moduli spaces of Riemann surfaces, both of which have been introduced in the previous chapter. After some introductory remarks on the construction of covers and Hurwitz spaces (Section 3.1), the focus is set on Hurwitz spaces of branched covers of a disk (Section 3.2), which we examine more closely in Chapters 5 and 6 with regard to homological stability. The chosen class of Hurwitz spaces behaves nicely in several ways (Section 3.3), giving rise to an H-space structure which is essential to our further examinations.

3.1. The classical construction of covers

The foundation for the study of branched covers of Riemann surfaces was laid in the second half of the 19th century. With articles by Lüroth [Lür71] and Clebsch [Cle73] as a foundation, Hurwitz investigated covers of $\mathbb{C}$ by Riemann surfaces with a given branch locus in his seminal article [Hur91]. With the aim of eventually counting the number of different covers with a given branch locus, he showed the bijection between isomorphism classes of m-fold holomorphic covers of $\mathbb{C}$ branched at $S \in \mathrm{Conf}_n(\mathbb{C})$ and tuples $\underline{g} = (g_1, \ldots, g_n) \in \mathcal{S}_m^n/\mathcal{S}_m$ (where $\mathcal{S}_m$ acts on n-tuples by simultaneous conjugation of the entries) such that

(i) $\partial\underline{g} = \prod_{i=1}^n g_i = 1$, and

(ii) the entries of $\underline{g}$ generate a transitive subgroup of $\mathcal{S}_m$,

known as *Hurwitz vectors* (note that we do not exclude trivial entries). This is (a simple version of) the *Riemann existence theorem*. For a thorough treatment of this theorem in its various forms, cf. [Fri04].

Given a branch locus S and a Hurwitz vector $\underline{g}$, Hurwitz explains the construction of the corresponding branched cover as follows: First, we join the points of S with $* \in \mathbb{C}\setminus S$ by a system of n non-intersecting arcs. One of the points of S is given the

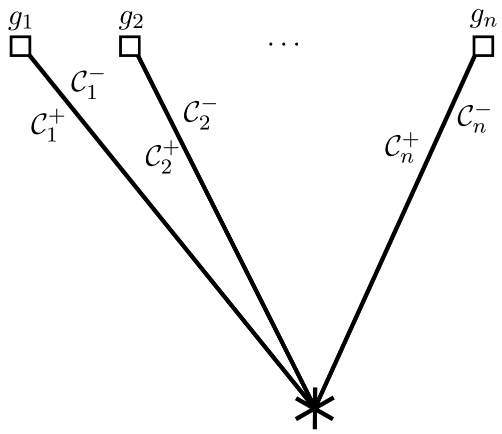


Figure 3.1.: The construction of a branched cover.

index 1, and we number the remaining points accordingly, employing the clockwise order of the arcs at $*$. This also yields an order on the chosen arcs $\mathcal{C}_1, \ldots, \mathcal{C}_n$. Now, we slice the complex plane along $\mathcal{C}_1, \ldots, \mathcal{C}_n$. For any $i = 1, \ldots, n$, we denote the first side (in clockwise order) of the slit at $\mathcal{C}_i$ by $\mathcal{C}_i^+$ and the second one by $\mathcal{C}_i^-$.

To construct the cover, take m ordered copies $\mathbb{C}_1, \ldots, \mathbb{C}_m$ of the sliced plane and glue $\mathcal{C}_i^+$ on $\mathbb{C}_i$ to $\mathcal{C}_i^-$ on $\mathbb{C}_{g_i(j)}$ in order to obtain an m-fold cover of $\mathbb{C}$ branched at S. Now, if we choose clockwise loops around the $\mathcal{C}_i$ as a basis for the group $\pi_1(\mathbb{C} \setminus S, 0) \cong F_n$, where we choose a base point $0 \in \mathbb{C} \setminus (S \cup \{*\})$, the monodromy around the i-th loop equals g_i.

Putting a different order on the sheets amounts to simultaneous conjugation of the entries of $\underline{g}$ by the corresponding permutation. Via the action of $\mathrm{Br}_n \cong \mathrm{Map}(\mathbb{C} \setminus S)$, any different system of arcs can be obtained from the original one up to isotopy. Modifying the arcs by the mapping class σ_i has the following effect on Hurwitz vectors:

$$\sigma_i \cdot \underline{g} = (g_1, \ldots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \ldots, g_n). \tag{3.1}$$

This is due to the work of Clebsch, cf. [Cle73]. The two conditions we put on the Hurwitz vector $\underline{g}$ ensure that (i) there is no branch point at infinity, and that (ii) the cover is connected.

In the case of simple branch points, i.e., only transpositions as entries of $\underline{g}$, Hurwitz showed that the parameter space $\mathcal{H}_{m,n}$ for simply n-branched m-fold covers of $\mathbb{C}$ carries the structure of a complex manifold. The space $\mathcal{H}_{m,n}$ admits an unbranched covering map to $\mathrm{Conf}_n(\mathbb{C})$, which is the *branch point map*. Moreover, the monodromy action of $\pi_1(\mathrm{Conf}_n(\mathbb{C})) \cong \mathrm{Br}_n$ on tuples $\underline{g}$ is given by the *Hurwitz action* described in (3.1). Here, somewhat unusually, we write the monodromy action as a *left* action such that it coincides with the mapping class action. Using this, Hurwitz was able to show the connectedness of $\mathcal{H}_{m,n}$ by proving that the Br_n-action on Hurwitz vectors with only transpositions as entries is transitive. This is equally true if we replace $\mathbb{C}$ by the sphere $\mathbb{P}^1_{\mathbb{C}}$. By using this, Severi showed in [Sev68] that the moduli space $\mathcal{M}_g$ is connected by proving that any Riemann surface of genus g admits a simply branched meromorphic function of degree n as long as $n \geq g+1$.

Hurwitz spaces also play an important role in arithmetics, where they are closely connected to the question of the existence of Galois groups isomorphic to a given finite group G. We postpone a short exposition of this correspondence to Remark 3.8.

3.2. Covers of disks

Let G be a finite group. For convenience, we replace $\mathbb{C}$ by a disk D in what follows, where we mark a point $* \in \partial D$.

In Section 3.1, we constructed an isomorphism class of m-fold branched covers $X \to D$ from a branch locus S and the monodromy $\mu\colon \pi_1(D \setminus S, *) \to \mathcal{S}_m$. The homomorphism μ may be applied to construct another cover: Let $G \subset \mathcal{S}_m$ be the image of μ. The kernel of μ is a normal subgroup of $\pi_1(D \setminus S, *)$ of index $|G|$, and by covering space theory, it defines a G-Galois cover of $D \setminus S$ up to isomorphism. In this sense, the original m-fold cover corresponds to a subgroup $\mu^{-1}(G') \subset \pi_1(D \setminus S, *)$, where G' is an index m subgroup of G.

If we pick a basis of $F_n \cong \pi_1(D \setminus S, *)$ and an identification of the fiber above $*$ with G, the monodromy of the G-Galois cover around the i-th loop can be identified with multiplication by $g_i \in G$. Note that this may be retranslated to the classical construction by choosing a suitable embedding $G \hookrightarrow \mathcal{S}_{|G|}$, so the statements of the Riemann existence theorem as well as the appearance of the Hurwitz action remain unaffected. For more details about the construction of branched covers from subgroups of $\pi_1(D \setminus S, *)$, cf. [FV91, §1.2] or the introduction to [CLP11].

3.2.1. Hurwitz spaces for n-marked branched G-covers

From now on, we focus on branched G-covers of D, where the covering spaces need not necessarily be connected. The following definition is, in essence, taken from [EVW16], making our results in Chapters 5 and 6 comparable to the ones in that article.

Definition 3.1. Let $n \in \mathbb{N}$. A *marked n-branched G-cover of D* is a set $S \in \mathrm{Conf}_n$ of branch points together with a homomorphism $\mu\colon \pi_1(D \setminus S, *) \to G$.

Remark 3.2. The kind of branched covers we just defined differs from the one considered by Hurwitz:

(i) An n-branched G-cover can be identified with the following topological notion, cf. [EVW16, Sect. 2.3]: A *geometric marked n-branched G-cover of D* is an isomorphism class of tuples $(Y, p, \bullet, S, \alpha)$, where $S \in \mathrm{Conf}_n$ is a branch locus, $p\colon Y \to D \setminus S$ a covering space map, $\alpha\colon G \to \mathrm{Aut}(p)$ a homomorphism into the deck transformation group of p inducing a free and transitive action of G on any fiber of p, and $\bullet \in Y$ a marked point in the fiber above $*$. Here, an *isomorphism* is a homeomorphism of the total spaces of the coverings which is compatible with the remaining data.

Given a geometric marked n-branched cover of D, we obtain a homomorphism $\mu\colon \pi_1(D \setminus S, *) \to G$ by the following construction: The choice of $\bullet$ together with the free and transitive action of G on $p^{-1}(*)$ yields a unique bijection between $p^{-1}(*)$ and G, choosing $\bullet = 1$. The monodromy action of $\pi_1(D \setminus S, *)$ on $p^{-1}(*)$ can then be identified with multiplication by elements in G.

Vice versa, by the Riemann existence theorem and the remarks at the beginning of this section, the choice of a branch locus S and a monodromy homomorphism $\mu\colon \pi_1(D \setminus S, *) \to G$ corresponds to an isomorphism class of unbranched G-covers of $D \setminus S$. The fact that we do not consider such homomorphisms up to conjugation in G is reflected by the choice of a marked point $\bullet$ in the fiber above $*$.

For more about the construction of branched G-covers, cf. also [RW06, Sect. 3.1].

(ii) If we choose a set of generators for $\pi_1(D \setminus S, *) \cong F_n$, we may identify the set of homomorphisms $\mu\colon \pi_1(D \setminus S, *) \to G$ with G^n. Hence, the homomorphism μ corresponds to a Hurwitz vector. Note that we do not put any restrictions on the set of homomorphisms: We allow branching at infinity as well as disconnected covering spaces.

In order to obtain a well-behaved model for the Hurwitz space for marked n-branched G-covers of D, we have to recall a few notions and facts from equivariant topology:

Definition 3.3. Let G be a group and X a space with a left H-action. The *Borel construction* of X modulo H is the space $EH \times_H X = (EH \times X)/H$, where H acts on $EH \times X$ from the right via $(e, x) \cdot h = (e \cdot h, h^{-1} \cdot x)$.

Remark 3.4. The topology on $EH \times_H X$ is induced by the topologies on EH and X. In fact, if H is discrete, the action of H on EH and hence on $EH \times X$ is free and properly discontinuous; so if X is a (possibly disconnected) manifold, $EH \times_H X$ is a (not necessarily connected) manifold as well.

Apart from that, $EH \times_H X \to EH/H = BH$ is the fiber bundle associated to the principal bundle $EH \to BH$ and the space X with H-action, cf. [Hus66, Ch. 4.5]. If X is discrete, $EH \times_H X \to BH$ is a cover of BH with fiber X. The connected components of the Borel construction are in bijective correspondence with the H-orbits in X.

Lemma 3.5. *Let H be a discrete group and $Y \to BH$ a covering space map with fiber X above $b \in BH$. Then Y is isomorphic as a covering space of BH to $EH \times_H X$, where the H-action on X is given by the (left) monodromy action of $\pi_1(BH, b) \cong H$.*

Proof. Clearly, we may write $EH \times_H X = \bigsqcup_{j \in J} EH \times_H X_j$, where $X = \bigsqcup_{j \in J} X_j$ is the decomposition of X into H-orbits. Hence, it suffices to prove the statement for a transitive H-action on X, i.e., Y connected.

Let $x \in X$. By covering space theory, the fundamental group of Y is isomorphic to the stabilizer subgroup H_x. Furthermore, a loop $h \in \pi_1(BH, b)$ lifts to a closed path in the Borel construction if and only if $(e, x) = (e \cdot h, x) \in EH \times_H X$, where e is a point above b in the universal cover $EH \to BH$. But this holds if and only if $h \cdot x = x$, so the fundamental group of $EH \times_H X$ is isomorphic to H_x as well. This implies that Y and $EH \times_H X$ must be isomorphic as covering spaces of BH (cf. [Hat02, Thm. 1.38]). $\square$

We may finally declare what kind of spaces we want to call *Hurwitz spaces*. By Definition 3.1, the space of marked n-branched G-covers must be an unbranched cover of $\mathrm{Conf}_n \simeq B\mathrm{Br}_n$ with fiber $\pi_1(D \setminus C_n, *) \cong G^n$ above C_n. The left monodromy action of $\mathrm{Br}_n \cong \pi_1(\mathrm{Conf}_n, C_n)$ on G^n is induced by the $\mathrm{Map}(D_n) \cong \mathrm{Br}_n$-action on generators of $\pi_1(D \setminus C_n)$. It is thus given by the Hurwitz action (3.1), cf. also Remark 2.10. Now, Lemma 3.5 yields the following definition:

Definition 3.6. Let G be a finite group, and $n \in \mathbb{N}$. The *Hurwitz space for marked n-branched G-covers of the disk* is defined as

$$\mathrm{Hur}_{G,n} = \widetilde{\mathrm{Conf}}_n \times_{\mathrm{Br}_n} G^n \simeq E\mathrm{Br}_n \times_{\mathrm{Br}_n} G^n,$$

where $\widetilde{\mathrm{Conf}}_n$ denotes the universal cover of Conf_n. Here, Br_n acts on G^n via the Hurwitz action (3.1). The *Hurwitz space for n-branched G-covers of the disk* is the quotient $\mathcal{H}_{G,n} = \mathrm{Hur}_{G,n}/G$, where G acts on $\mathrm{Hur}_{G,n}$ by simultaneous conjugation of Hurwitz vectors.

Remark 3.7. $\mathrm{Hur}_{G,0} = \mathcal{H}_{G,0}$ is a single point with trivial G-action, corresponding to the trivial $|G|$-fold cover of D. This coincides with the above definition if we set $\mathrm{Br}_0 = 1$ to be the trivial group and G^0 a single element.

The definition of the Hurwitz space for unmarked covers is due to the fact that, given a basis of $\pi_1(D \setminus S, *)$, two Hurwitz vectors define the same cover up to the choice of a marked point in the fiber if and only if they can be mapped to each other via simultaneous conjugation by an element in G.

Remark 3.8. Hurwitz spaces are closely connected to the *Regular Inverse Galois Problem*: Given a finite group G, is there a $\mathbb{Q}$-regular Galois extension of $\mathbb{Q}(x)$ with n branch points? In terms of Hurwitz *schemes*, this translates to the question whether $\mathcal{H}_{G,n}$ as a Hurwitz *scheme* has a $\mathbb{Q}$-rational point, cf. [FV91, §0].

That being said, we will focus on the topology of Hurwitz spaces in the following. More about the algebraic construction of Hurwitz schemes as well as the arithmetic applications can be found in the survey article [RW06]. For an introduction to the (Regular) Inverse Galois Problem, we refer to the monographs [Völ96] and [MM99].

3.2.2. Combinatorial invariants and subspaces

If the sequence $\{\mathrm{Hur}_{G,n} \mid n \geq 0\}$ were homologically stable, there would be an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $b_0(\mathrm{Hur}_{G,N}) = b_0(\mathrm{Hur}_{G,n})$ for the zeroth Betti numbers. This section is about invariants that are constant on connected components of Hurwitz spaces; we will eventually see that the spaces $\mathrm{Hur}_{G,n}$ are usually too large to give hope for a homological stability theorem. Instead, we specify a sequence of suitable subspaces whose associated sequence of zeroth Betti numbers is not *a priori* a strictly increasing function in n.

We start with some easy, yet important consequences of the definition of $\mathrm{Hur}_{G,n}$:

Proposition 3.9. *Let $n \in \mathbb{N}$. We have:*

(i) The connected components of $\mathrm{Hur}_{G,n}$ are in natural bijective correspondence to the Br_n-orbits in G^n.

(ii) The component of $\mathrm{Hur}_{G,n}$ corresponding to the orbit of a Hurwitz vector $\underline{g} \in G^n$ is of type $K(\pi, 1)$, where π is isomorphic to the stabilizer subgroup $(\mathrm{Br}_n)_{\underline{g}}$ for the Hurwitz action of Br_n on G^n.

Proof. From the definition of $\mathrm{Hur}_{G,n}$ as a finite cover of $B\mathrm{Br}_n$, these results are almost immediate: (i) is clear from Remark 3.4, while (ii) follows from the asphericity of $B\mathrm{Br}_n$

and the fact that the fundamental group of a connected covering space is isomorphic to the stabilizer under the monodromy action of a point in the fiber. $\square$

We will now define some Br_n-invariant functions on Hurwitz spaces. It follows from Proposition 3.9(i) that such invariants are constant on connected components of $\mathrm{Hur}_{G,n}$.

Note that a pair $(S, \underline{g}) \in B\mathrm{Br}_n \times G^n$ defines a unique branched cover up to the choice of an isomorphism $\pi_1(D \setminus S, *) \to F_n$, that is, up to the action of $\mathrm{Br}_n \cong \mathrm{Map}(D \setminus S)$ on a set of generators for $\pi_1(D \setminus S, *)$. Since we define Br_n-invariant quantities, they do not depend on this choice. For now, we let $(S, \underline{g})$, with $\underline{g} = (g_1, \dots, g_n) \in G^n$, be a fixed isomorphism class of marked n-branched covers of S.

- The *global monodromy* of $(S, \underline{g})$ is the subgroup of G generated by $g_1, \dots, g_n$. Geometrically, it is equal to G if and only if the corresponding branched cover is connected. In this case, we say that the branched cover has *full monodromy.*
- The *boundary monodromy* of $(S, \underline{g})$ is defined as the product $\partial \underline{g} = \prod_{i=1}^n g_i$. Its inverse describes the branching behavior *at infinity.*
- The *Nielsen class* of $(S, \underline{g})$ is the multiset $\nu(\underline{g}) = \{c_{j_1}, \dots, c_{j_n}\}$, where the $c_i \subset G$ are conjugacy classes and $g_i \in c_{j_i}$ for all $i = 1, \dots, n$. Given conjugacy classes $c_1, \dots, c_t$ in $\nu(\underline{g})$, the *shape (vector)* of $(S, \underline{g})$ is the t-tuple $(n_{c_1}(\underline{g}), \dots, n_{c_t}(\underline{g}))$, where $n_{c_i}(\underline{g})$ is the number of elements of $\underline{g}$ that lie in c_i.

There are cases in which the global and boundary monodromy as well as the shape (or the Nielsen class, respectively) form a complete set of invariants for the determination of connected components of $\mathrm{Hur}_{G,n}$, cf. Theorem 6.23.

Let G be a group with two conjugacy classes c_1, c_2. Then, for covers with n branch points, there are $n+1$ different possible shapes $(n-i, i)$, $i = 0, \dots, n$. Therefore, there is a lower bound $b_0(\mathrm{Hur}_{G,n}) \geq n+1$ for any group, unless G is trivial (which is the case $\mathrm{Hur}_{G,n} \simeq \mathrm{Conf}_n$). This implies that in nontrivial cases, the sequence $\{\mathrm{Hur}_{G,n} \mid n \geq 0\}$ cannot satisfy any form of homological stability. Therefore, it makes sense to consider subspaces corresponding to fixed shapes.

From now on, let $c = (c_1, \dots, c_t)$ be a tuple of t distinct conjugacy classes in G. For a shape vector $\underline{\xi} = (\xi_1, \dots, \xi_t) \in \mathbb{N}_0^t$ of length $\xi = \sum_{i=1}^t \xi_i$, we consider the subspace $\mathrm{Hur}^c_{G,\underline{\xi}}$ of $\mathrm{Hur}_{G,\xi}$ which parametrizes covers with shape $\underline{\xi}$. By what we have seen above, it is a union of connected components of $\mathrm{Hur}_{G,\xi}$.

The space $\mathrm{Hur}^c_{G,\underline{\xi}}$ is a cover of Conf_ξ with the tuples in G^n for which ξ_i entries lie in c_i as a fiber. This cover factors over $\mathrm{Conf}_{\underline{\xi}}$, which is the colored configuration space introduced in Section 2.2.2. The fiber of the unbranched cover $\mathrm{Hur}^c_{G,\underline{\xi}} \to \mathrm{Conf}_{\underline{\xi}}$ can be identified with

$$\mathbf{c} = c_1^{\xi_1} \times \ldots \times c_t^{\xi_t},$$

on which $\mathrm{Br}_{\underline{\xi}}$ acts via the Hurwitz action (note that $\mathbf{c}$ depends on the choices of c and $\underline{\xi}$). Now, Lemma 3.5 justifies the construction of $\mathrm{Hur}^c_{G,\underline{\xi}}$ we make in the following.

Furthermore, it makes sense to consider spaces which parametrize only connected covers, i.e., covers with full monodromy. We denote by G^n_{gen} the subset of G^n of Hurwitz vectors whose entries generate G.

Definition 3.10. Using the above notation, we define the following subspaces of $\mathrm{Hur}_{G,\xi}$:

- $\mathrm{CHur}_{G,\xi} \simeq E\mathrm{Br}_\xi \times_{\mathrm{Br}_\xi} G^\xi_{\mathrm{gen}}$
- $\mathrm{Hur}^c_{G,\underline{\xi}} \simeq E\mathrm{Br}_{\underline{\xi}} \times_{\mathrm{Br}_{\underline{\xi}}} \mathbf{c}$
- $\mathrm{CHur}^c_{G,\underline{\xi}} \simeq E\mathrm{Br}_{\underline{\xi}} \times_{\mathrm{Br}_{\underline{\xi}}} \left(\mathbf{c} \cap G^\xi_{\mathrm{gen}}\right)$

In the following, we will focus on the sequence $\{\mathrm{Hur}^c_{G,n\cdot\underline{\xi}} \mid n \geq 0\}$, where we write $n \cdot \underline{\xi}$ for the shape $(n\xi_1, \ldots, n\xi_t)$. We call this sequence the *diagonal direction* for the shape $\underline{\xi}$. In Chapters 5 and 6, we will see that under suitable extra conditions, this sequence satisfies homological stability.

3.3. Structure on Hurwitz spaces

In this section, we will begin to set the stage for the homological investigation in Chapter 5, introducing the *ring of connected components.* We fix a finite group G, a tuple $c = (c_1, \ldots, c_t)$ of distinct conjugacy classes in G, and a shape vector $\underline{\xi} \in \mathbb{N}^t$.

Before we start, it is good to know that we may work in the category of CW complexes. This follows from the fact that Hurwitz spaces are homotopy equivalent to finite covers of configuration spaces, which are known to have the structure of CW complexes, cf. the proof of [EVW16, Prop. 2.5] and references therein.

Proposition 3.11. *For all $n \in \mathbb{N}$, the spaces $\mathrm{Hur}_{G,n}$ (and thus all of their connected components) are homotopy equivalent to CW complexes with finitely many cells.*

The existence of the H-space structure on the union of Hurwitz spaces $\mathrm{Hur}_{G,n}$ is stated in [EVW16, Sect. 2.6]. We enclose a detailed proof of the statement below:

Proposition 3.12. *For all shapes $\underline{\mu}, \underline{\nu} \in \mathbb{N}_0^t$, there are continuous maps*

$$\mathrm{Hur}^c_{G,\underline{\mu}} \times \mathrm{Hur}^c_{G,\underline{\nu}} \to \mathrm{Hur}^c_{G,\underline{\mu}+\underline{\nu}},$$

defined and associative up to homotopy, which are compatible to the natural monomorphism

$$\mathrm{Br}_\mu \times \mathrm{Br}_\nu \to \mathrm{Br}_{\mu+\nu} \tag{3.2}$$

and the obvious bijection

$$G^\mu \times G^\nu \to G^{\mu+\nu}. \tag{3.3}$$

For a fixed tuple $\underline{\xi} \in \mathbb{N}_0^t$, these maps give $\coprod_{n\geq 0} \mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ the structure of a disconnected H-space with homotopy identity $\mathrm{Hur}^c_{G,0\cdot\underline{\xi}}$.

Proof. Let $\underline{g} \in G^\mu, \underline{h} \in G^\nu$ be Hurwitz vectors with respective shapes $\underline{\mu}$ and $\underline{\nu}$. Now, (3.2) restricts to a map of stabilizer subgroups $(\mathrm{Br}_\mu)_{\underline{g}} \times (\mathrm{Br}_\nu)_{\underline{h}}$ with image in $(\mathrm{Br}_{\mu+\nu})_{(\underline{g},\underline{h})}$, where we use (3.3) to produce the vector $(\underline{g},\underline{h}) \in G^{\mu+\nu}$ of shape $\underline{\mu}+\underline{\nu}$.

$$(\mathrm{Br}_\mu)_{\underline{g}} \times (\mathrm{Br}_\nu)_{\underline{h}} \to (\mathrm{Br}_{\mu+\nu})_{(\underline{g},\underline{h})} \tag{3.4}$$

defines a unique free homotopy class of maps $B(\mathrm{Br}_\mu)_{\underline{g}} \times B(\mathrm{Br}_\nu)_{\underline{h}} \to B(\mathrm{Br}_{\mu+\nu})_{(\underline{g},\underline{h})}$, where we employ the isomorphism $(\mathrm{Br}_\mu)_{\underline{g}} \times (\mathrm{Br}_\nu)_{\underline{h}} \cong \pi_1(B(\mathrm{Br}_\mu)_{\underline{g}} \times B(\mathrm{Br}_\nu)_{\underline{h}})$.

By Proposition 3.9(ii), $B(\mathrm{Br}_\mu)_{\underline{g}}$ and $B(\mathrm{Br}_\nu)_{\underline{h}}$ are homotopy equivalent to connected components $X_{\underline{g}} \subset \mathrm{Hur}^c_{G,\underline{\mu}}$ and $X_{\underline{h}} \subset \mathrm{Hur}^c_{G,\underline{\nu}}$, respectively. It remains to show that Hurwitz-equivalent choices of $\underline{g}$ and $\underline{h}$ yield the same map, up to homotopy.

Let $\underline{g}' = \beta_1\underline{g}$ and $\underline{h}' = \beta_2\underline{h}$ for some $\beta_1 \in \mathrm{Br}_\mu$ and $\beta_2 \in \mathrm{Br}_\nu$. Now, $\underline{g}$ and $\underline{g}'$ correspond to different choices of base points in the fiber of $X_{\underline{g}} \to B\mathrm{Br}_\mu \simeq \mathrm{Conf}_\mu$ above C_μ (accordingly for $\underline{h}, \underline{h}'$). Changing the base point of a map of classifying spaces corresponds to conjugation of Hurwitz vectors. Therefore, the independence of the choice of $\underline{g}$ and $\underline{h}$ follows directly from the commutativity of the diagram

$$\begin{tikzcd}
(\mathrm{Br}_\mu)_{\underline{g}} \times (\mathrm{Br}_\nu)_{\underline{h}} \arrow[r] \arrow[d, "(\,)^{\beta_1^{-1}} \times (\,)^{\beta_2^{-1}}"] & (\mathrm{Br}_{\mu+\nu})_{(\underline{g},\underline{h})} \\
(\mathrm{Br}_\mu)_{\underline{g}'} \times (\mathrm{Br}_\nu)_{\underline{h}'} \arrow[r] & (\mathrm{Br}_{\mu+\nu})_{(\underline{g}',\underline{h}')}, \arrow[u, "((\,)^{\beta_1},(\,)^{\beta_2})"']
\end{tikzcd}$$

where $\left((\,)^{\beta_1},(\,)^{\beta_2}\right)$ denotes conjugation by the image of (β_1,β_2) under the inclusion (3.2), and the horizontal arrows are the maps from (3.4). Associativity (up to homotopy) of the multiplication maps follows from the associativity of the maps (3.2) and (3.3) for multiple factors.

Combining these maps, we obtain maps $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}} \times \mathrm{Hur}^c_{G,m\cdot\underline{\xi}} \to \mathrm{Hur}^c_{G,(n+m)\cdot\underline{\xi}}$, defined up to homotopy, which give $\coprod_{n\geq 0} \mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ the structure of an H-space, where the single point of $\mathrm{Hur}^c_{G,0\cdot\underline{\xi}}$ is the unique (homotopy) identity. □

Remark 3.13. Geometrically, the multiplicative structure on branched covers of D stated in Proposition 3.12 can be understood as follows, cf. [EVW16, Sect. 2.6]: Choose two disjoint embedded loops γ_1, γ_2 in D based at $*$. These loops enclose disks Y_1, Y_2. For two marked branched G-covers $(S, \mu) \in \mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ and $(S', \mu') \in \mathrm{Hur}^c_{G,m\cdot\underline{\xi}}$, we may understand (S, μ) as a branched cover of Y_1 and (S', μ') as a branched cover of Y_2. The product of the covers can be thought of as the extension to D of these covers of Y_1 and Y_2 by the trivial G-cover on the complement, where we need to make sure that the choice of $\bullet$ in the fiber above $*$ is consistent, cf. also Remark 3.2(i).

In the following, let A be a commutative ring. We will now make a first step towards the homology of Hurwitz spaces: The H-space structure on the union of Hurwitz spaces induces a ring structure on the zeroth homology of this union:

Definition 3.14. The A-module

$$R^{A,c}_{G,\underline{\xi}} = \bigoplus_{n\geq 0} H_0(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; A)$$

is called the *ring of connected components (with coefficient ring A)* for the sequence $\{\mathrm{Hur}^c_{G,n\cdot\underline{\xi}} \mid n \geq 0\}$ of Hurwitz spaces. If G, $\underline{\xi}$, A, and c are clear from the context, we simply denote the ring by R.

Remark 3.15. The continuous multiplication from Proposition 3.12 provides R with a graded ring structure (grading in the n-variable).

There is a nice combinatorial description of the ring R, cf. [FV91] and [EVW16]: Let $\mathfrak{s} = \coprod_{n\geq 0} \mathbf{c}^n / \mathrm{Br}_{n\cdot\underline{\xi}}$ be the set of Hurwitz vectors of any length, up to the Hurwitz action. Here, note that we identify $\mathrm{Br}_{n\cdot\underline{\xi}} \subset \mathrm{Br}_{n\xi}$ as the set stabilizer of $\mathbf{c}^n \subset G^n$. Then, concatenation of Hurwitz vectors gives $\mathfrak{s}$ the structure of a monoid with the empty tuple as the identity. Applying Propositions 3.9(i) and 3.12, we see that R is the monoid algebra $A[\mathfrak{s}]$.

In particular, R is finitely generated as an A-algebra: Its degree one part is generated as an A-module by elements $r(g)$ with $g \in \mathbf{c} / \mathrm{Br}_{\underline{\xi}}$. Any element of $\mathbf{c}^n$ is the concatenation of n elements in $\mathbf{c}$, and this concatenation descends to a map $(\mathbf{c} / \mathrm{Br}_{\underline{\xi}})^n \to \mathbf{c}^n / \mathrm{Br}_{n\cdot\underline{\xi}}$ by virtue of the inclusion $(\mathrm{Br}_{\underline{\xi}})^n \to \mathrm{Br}_{n\cdot\underline{\xi}}$, cf. also (3.2). Therefore, the degree one elements $r(g)$ generate R as an A-algebra.

Notation 3.16. By misuse of notation, if $\underline{v} \in \mathbf{c}^n$ is a Hurwitz vector specifying a component $X \subset \mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$, we write $\underline{v} \in R$ for the element $1 \in A \cong H_0(X; A) \subset R$.

Lemma 3.17. *If $\underline{v} \in \mathbf{c}^n$ satisfies $\partial\underline{v} = 1$, it defines a central element of R.*

Proof. By Remark 3.15, we need to show that for any $\underline{g} \in \mathbf{c}^m$, the Hurwitz vectors $(\underline{v}, \underline{g})$ and $(\underline{g}, \underline{v})$ are equivalent under the $\mathrm{Br}_{(m+n)\cdot\underline{\xi}}$-action. The braid

$$\sigma = \prod_{i=1}^{m} \prod_{j=1}^{n} \sigma_{m-i+j}$$

pulls $\underline{v}$ through the tuple $\underline{g}$, resulting in

$$\sigma \cdot \left(\underline{v}, \underline{g}\right) = \left(\partial v g_1 (\partial v)^{-1}, \ldots, \partial v g_{m\xi} (\partial v)^{-1}, \underline{v}\right) = (\underline{g}, \underline{v}),$$

which proves the claim. □

We are interested in the homology modules of $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$. So far, we considered the zeroth homology modules and obtained the ring R from the H-space structure of the union of Hurwitz spaces. There is also an extra structure on higher homology modules which we will see in what follows.

By the Künneth formula (cf. [Spa66, Thm. 5.3.10]) and the H-space structure on Hurwitz spaces, we have a unique map

$$H_0(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; A) \times H_p(\mathrm{Hur}^c_{G,m\cdot\underline{\xi}}; A) \to H_p(\mathrm{Hur}^c_{G,(n+m)\cdot\underline{\xi}}; A) \tag{3.5}$$

for all integers $m, n, p \in \mathbb{N}_0$. In particular, this gives the direct sum of the p-th homology modules of Hurwitz spaces the structure of a graded R-module.

Notation 3.18. We denote the graded $R^{A,c}_{G,\underline{\xi}}$-module (grading in the n-variable) described above by

$$M^{A,c}_{G,\underline{\xi},p} = \bigoplus_{n\geq 0} H_p(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; A),$$

or, if no misunderstandings are possible, by M_p. Clearly, we have $M_0 = R$.

With the help of the introduced notation, we might rephrase homological stability for the sequence $\{\mathrm{Hur}^c_{G,n\cdot\underline{\xi}} \mid n \geq 0\}$ in terms of M_p. In fact, our strategy in Chapter 5 will be to find a suitable homogeneous element $U \in R$ such that multiplication by U, $M_p \xrightarrow{U\cdot} M_p$, is an isomorphism in high degrees. The existence of such an element implies $(\deg U)$-periodic homological stability.

4. Plants

In the introduction to Chapter 2, we portrayed the significance of highly connected simplicial complexes for homological stability proofs. As the Hurwitz spaces $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ are finite covers of the colored configuration spaces $\mathrm{Conf}_{n\cdot\underline{\xi}} \simeq B\mathrm{Br}_{n\cdot\underline{\xi}}$, we are looking for a highly connected simplicial complex with a well-behaved $\mathrm{Br}_{n\cdot\underline{\xi}}$-action on its simplices. The prior homological stability theorem for Hurwitz spaces in [EVW16] is proved with the help of a Br_n-action on the *arc complex*. The colored braid group $\mathrm{Br}_{n\cdot\underline{\xi}}$ acts on the arc complex as well, but the unfavorable structure of the occurring stabilizers calls for the definition of another class of simplicial complexes. The *colored plant complexes* are designed specifically for the work with all kinds of colored braid groups.

In Section 4.1, we define *plants* and *plant complexes* on open surfaces and prove high connectivity. After that, we focus on the more special *colored plant complexes* in Section 4.2. Though the $\mathrm{Br}_{n\cdot\underline{\xi}}$-action on q-simplices of these complexes is generally not transitive (not even on vertices), they turn out to be the right objects for our work with colored braid groups in Chapter 5.

Relevant definitions and facts for simplicial complexes are collected in Appendix A.

4.1. Plant complexes

Let S be a surface with non-empty boundary. In the following, we define plants in S as well as plant complexes. Our definition of plants is a generalizations of the *ferns* in [Tra14]. The feeling that a word like *Pteridophyta* (a more specific botanical generalization of ferns) is rather unpleasant let us stick with the name *plant*.

4.1.1. Plants and plant complexes

Let Δ be a finite set of δ points in S°, partitioned as $\Delta = \Delta_1 \sqcup \ldots \sqcup \Delta_t$, and $\delta_i \coloneqq |\Delta_i|$ for $i = 1, \ldots, t$; thus $\delta = \sum_{i=1}^t \delta_i$. We write $\underline{\delta} = (\delta_1, \ldots, \delta_t)$ and call Δ an *instance* for $\underline{\delta}$. Let $*$ be a base point in the boundary of S. An *arc* is a smooth embedding

$\gamma\colon I \to S$ with $\gamma(0) = *$ and $\gamma(1) \in \Delta$, meeting the boundary transversally, and with interior entirely in $S \setminus (\partial S \cup \Delta)$. We usually identify γ with the set $\gamma(I) \subset D$.

Definition 4.1. Let $\underline{\xi} = (\xi_1, \ldots, \xi_t) \in \mathbb{N}^t$ and $\xi = \xi_1 + \ldots + \xi_t$.

(i) A *$\underline{\xi}$-plant* in (S, Δ) is an unordered ξ-tuple of arcs in S which only meet at $*$, where for some permutation $\sigma \in \mathcal{S}_t$, exactly ξ_i arcs end at points of $\Delta_{\sigma(i)}$, for $i = 1, \ldots, t$. The tuple $\underline{\xi}$ is called the *pattern*.

(ii) A *colored $\underline{\xi}$-plant* in (S, Δ) is a $\underline{\xi}$-plant in (S, Δ) with the requirement that for $i = 1, \ldots t$, exactly ξ_i arcs end at points of Δ_i.

(iii) Two $\underline{\xi}$-plants v, w in (S, Δ) are called *equivalent* if there is an isotopy of S fixing $\partial S \cup \Delta$ pointwise that transforms one plant into the other. If v and w are equivalent, we write $v \sim_{\mathcal{P}} w$ (where the $\mathcal{P}$ indicates *plant*-equivalence).

(iv) For any plant u, we write $u^\circ = u \setminus (\{*\} \cup \Delta)$ for its *interior*.

(v) Certain plants have specific names:

- a *cofern* is a $(1, \ldots, 1)$-plant,
- an *(m, r)-fern* or *multifern* is a $(\underbrace{r, \ldots, r}_{m \text{ times}}, 0, \ldots 0)$-plant, where $\underline{\delta} = (r, \ldots, r)$.

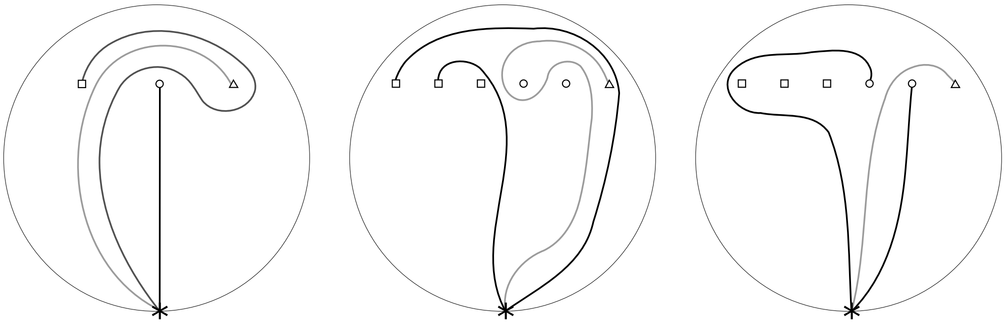
We note that $(1, r)$-ferns are *ferns* in the sense of [Tra14].

(vi) We say that two plants v, w (not necessarily of the same pattern) have *s points of intersection* if s is the minimal number such that there are plants $v' \sim_{\mathcal{P}} v$ and $w' \sim_{\mathcal{P}} w$ such that v'° and w'° share s points in $D^\circ \setminus \Delta$. We write $v.w = s$. For $v.w = 0$, we call v and w *disjoint*.

The definition of the intersection number of arcs is analogous (in fact, arcs are $(1, 0, \ldots, 0)$-plants).

First examples of plants in D can be seen in Figure 4.1. Given $\underline{\delta}$ and $\underline{\xi}$ as in the captions, the left and right plants are colored. Changing $\underline{\xi}$ to $(2, 0, 1)$, the middle plant is colored as well. For the intersection of plants, we have the following lemma:

Lemma 4.2. *Let $v = (a_1, \ldots, a_\zeta)$ and $w = (b_1, \ldots, b_\xi)$ be plants in (S, Δ) with arbitrary patterns. The intersection number $v.w$ is finite and arcwise distributive, i.e., we have $v.w = \sum_{i=1}^{\zeta} \sum_{j=1}^{\xi} a_i.b_j$.*



$\underline{\delta} = \underline{\xi} = (1, 1, 1)$ $\qquad$ $\underline{\delta} = (3, 2, 1)$, $\underline{\xi} = (0, 2, 1)$ $\qquad$ $\underline{\delta} = (3, 2, 1)$, $\underline{\xi} = (0, 2, 1)$

Figure 4.1.: Examples of $\underline{\xi}$-plants in a disk.

Proof. For the first part, it suffices to show that generically, two arcs a_1, b_1 meet in finitely many points. By the transversality theorem (cf. [Tho54]), the space of smooth embeddings $b_1 \colon I \to S$ which are transversal to a_1 is dense in the space of all smooth embeddings. Now, $a_1 \colon I \to S$ and $b_1 \colon I \to S$ being transversal implies that $b_1^{-1}(a_1(I)) \subset I$ is a 0-dimensional submanifold. Such a submanifold necessarily consists of only finitely many points.

Now, the inequality '$\geq$' in the second part of the lemma is clear by definition of the products $a_i.b_j$. By the first part of the proof, we may assume that all intersections of v and w are transversal. For the other inequality, we assume that v, w are in minimal position, i.e., $v.w = |v^\circ \cap w^\circ|$, and that we have $v.w > \sum_{i=1}^{\zeta} \sum_{j=1}^{\xi} a_i.b_j$.

By assumption, there exist indices p, q with

$$|a_p^\circ \cap b_q^\circ| > a_p.b_q. \tag{4.1}$$

Therefore, there must be segments of a_p and b_q that form a continuous loop. Choose k and l among all such p, q such that such a loop has no intersection with further arcs of v or w: By finiteness of the intersection number, there cannot be an infinite sequence of nested loops, so such indices exist. It follows from the Jordan curve theorem (cf. [Jor91]) that there is a closed disk $D_0 \subset S$ bounded by segments of a_k and b_l, containing no other arc segments of v or w.

After eventual slight smooth deformations of a_k or b_l, we may assume that neither $*$ nor Δ share points with D_0. By transversality of the intersections of v and w, these deformed arcs a'_k, b'_l can be chosen such that for the plants v', w' defined by replacing a_k by a'_k and b_k by b'_k, respectively, we have

$$|v'^\circ \cap w'^\circ| \leq |v^\circ \cap w^\circ| + 1. \tag{4.2}$$

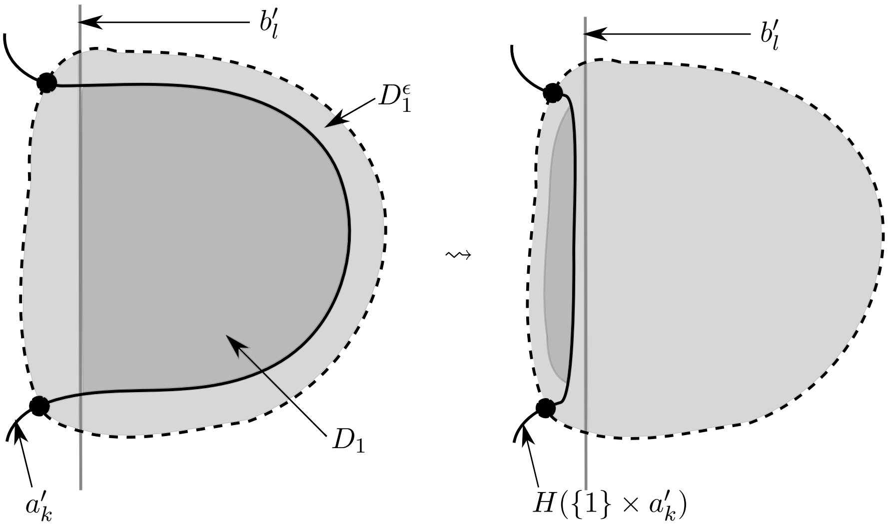


Figure 4.2.: The isotopy H from the proof of Lemma 4.2.

Indeed, if both $*$ and Δ intersected D_0, there would be no interior point of intersection between a_k and b_l, contradicting (4.1). Denote by D_1 the closed disk bounded by segments of a'_k and b'_l.

We now define an isotopy $H\colon I \times S \to S$ that satisfies

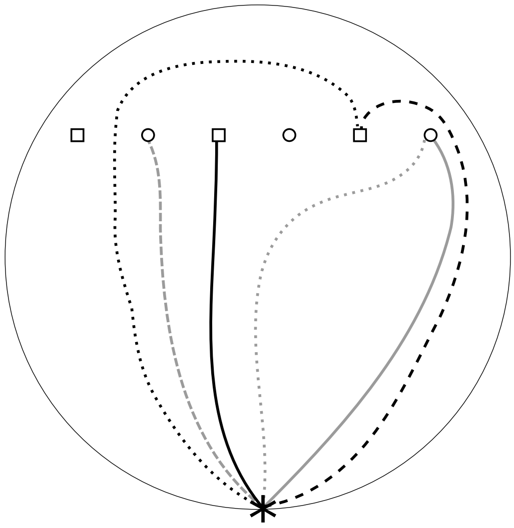
$$|H(\{1\} \times {a'}^{\circ}_{k}) \cap {b'}^{\circ}_{l}| < |{a'}^{\circ}_{k} \cap {b'}^{\circ}_{l}|, \tag{4.3}$$

$$|H(\{1\} \times {v'}^{\circ}) \cap {w'}^{\circ}| \leq |{v'}^{\circ} \cap {w'}^{\circ}| - 2. \tag{4.4}$$

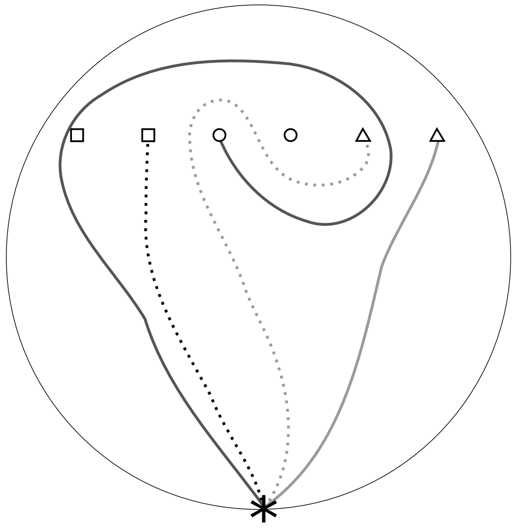
Let $\epsilon > 0$ and D_1^ϵ an open ϵ-neighborhood of D_1, where we choose ϵ such that there are no segments of arcs in D_1^ϵ other than a'_k and b'_l, and such that D_1^ϵ lies entirely in the interior of S. Now, the arc segment $a'_k \cap \overline{D_1^\epsilon}$ is isotopic (fixing endpoints) to an arc segment in $\overline{D_1^\epsilon}$ which does not intersect $b'_l \cap \overline{D_1^\epsilon}$. By the isotopy extension theorem (cf. [Pal60]), this isotopy may be extended to an ambient isotopy $h\colon I \times \overline{D_0^\epsilon} \to \overline{D_1^\epsilon}$ which fixes the boundary circle $\partial\overline{D_1^\epsilon}$ pointwise. We extend h by the identity on $S \setminus D_1^\epsilon$ and denote the resulting isotopy of D by H.

Now, (4.3) is satisfied since we push D_1 across b'_l and thus remove two intersections. As the potential slight deformation of a'_k and b'_l creates at most one extra intersection, the application of H removes at least one intersection point. Then, (4.4) follows from the choice of ϵ.

From (4.2) and (4.4), we obtain $|H(\{1\} \times {v'}^{\circ}) \cap {w'}^{\circ}| < |v^{\circ} \cap w^{\circ}| = v.w$, which contradicts the definition of the intersection number. The assertion follows. $\square$



2-simplex in $\mathrm{F}\mathcal{P}^{(1,1)}_{(3,3)}(D)$



1-simplex in $\mathcal{P}^{(1,1,0)}_{(2,2,2)}(D)$

Figure 4.3.: Representatives of simplices in plant complexes (1) – colors indicate the type of endpoints, line styles distinguish between different plants.

Remark 4.3. Clearly, for disjoint plants, there exist equivalent plants whose interiors are not disjoint as sets. From now on, if possible, we assume without further notification that for any two isotopy classes of arcs/plants, we choose representatives such that their number of intersections is minimal.

Definition 4.4. With notation as above, we define:

(i) The *full $\underline{\xi}$-plant complex* $\mathrm{F}\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)$ is the simplicial complex with isotopy classes of $\underline{\xi}$-plants in (S, Δ) as vertices. A q-simplex in $\mathrm{F}\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)$ is a set of $q+1$ isotopy classes of $\underline{\xi}$-plants on (S, Δ) which can be embedded with disjoint interiors.

(ii) The *$\underline{\xi}$-plant complex* $\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)$ is the subcomplex of $\mathrm{F}\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)$ which contains the simplices $\alpha \in \mathrm{F}\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)$ such that no two plants of α share a point in Δ.

(iii) The *full colored $\underline{\xi}$-plant complex* $\mathrm{F}\mathcal{O}^{\underline{\xi}}_{\underline{\delta}}(S) \subset \mathrm{F}\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)$ and the *colored $\underline{\xi}$-plant complex* $\mathcal{O}^{\underline{\xi}}_{\underline{\delta}}(S) \subset \mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)$ are the subcomplexes defined by the restriction that only colored plants are allowed as vertices.

Remark 4.5. For any two instances Δ, Δ' of $\underline{\delta}$, there is an isotopy of S that fixes ∂S pointwise and conveys one instance into the other. Thus, plant complexes on (S, Δ) and (S, Δ') are isomorphic as simplicial complexes, so the chosen notation makes sense.

Also, the choice of the marked point $* \in \partial S$, by connectedness of S, is irrelevant to the isomorphism type of plant complexes.

By definition, we have the following diagram of inclusions of simplicial complexes:

$$\begin{array}{ccc} \mathcal{O}^{\underline{\xi}}_{\underline{\delta}}(S) & \subset & \mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S) \\ \cap & & \cap \\ \mathrm{F}\mathcal{O}^{\underline{\xi}}_{\underline{\delta}}(S) & \subset & \mathrm{F}\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S) \end{array}$$

We introduce two partial orderings of $\mathbb{N}_0^t$:

- $(x_1, \ldots, x_t) \preccurlyeq (y_1, \ldots, y_t)$ if there is a permutation $\sigma \in \mathcal{S}_t$ such that $x_i \leq y_{\sigma(i)}$ for all $i = 1, \ldots, t$, and

- $(x_1, \ldots, x_t) \leq (y_1, \ldots, y_t)$ if $x_i \leq y_i$ for all $i = 1, \ldots, t$.

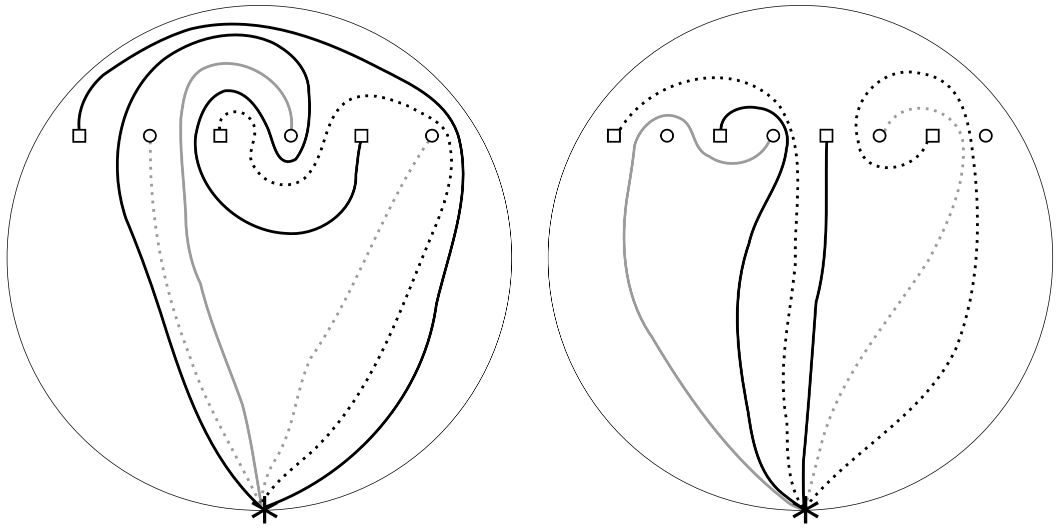
Immediately from the definitions, we obtain:

Lemma 4.6. *Both* $\mathrm{F}\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)$ *and* $\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)$ *are non-empty if and only if* $\underline{\xi} \preccurlyeq \underline{\delta}$*, and* $\mathrm{F}\mathcal{O}^{\underline{\xi}}_{\underline{\delta}}(S)$ *and* $\mathcal{O}^{\underline{\xi}}_{\underline{\delta}}(S)$ *are non-empty if and only if* $\underline{\xi} \leq \underline{\delta}$.

Proof. Non-emptiness of $\mathrm{F}\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)$ and $\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)$ ($\mathrm{F}\mathcal{O}^{\underline{\xi}}_{\underline{\delta}}(S)$ and $\mathcal{O}^{\underline{\xi}}_{\underline{\delta}}(S)$) is equivalent to the existence of a (colored) $\underline{\xi}$-plant. By connectedness of S and the fact that the points of Δ lie in the interior of S, this reduces to a purely combinatorial question about $\underline{\xi}$ and $\underline{\delta}$ which is solved by the condition $\underline{\xi} \preccurlyeq \underline{\delta}$ ($\underline{\xi} \leq \underline{\delta}$). $\square$

Remark 4.7. There is a natural total order on the vertices of simplices in $\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)$ ($\mathcal{O}^{\underline{\xi}}_{\underline{\delta}}(S)$) which induces the structure of an ordered simplicial complex: By imposing a Riemannian structure on S, we may assume that all arcs are parametrized by arc length. Now for plants v, w of the same pattern, we write $v < w$ if and only if there is an arc in v whose inward pointing unit tangent vector at $*$ occurs in clockwise order before any inward pointing unit tangent vector of an arc in w.

Due to the possible double occupancy of endpoints in $\mathrm{F}\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)$, there is no such order on full plant complexes.



1-simplex in $\mathcal{P}^{(2,1)}_{(3,3)}(D)$ 1-simplex in $\mathcal{O}^{(2,1)}_{(4,4)}(D)$

Figure 4.4.: Representatives of simplices in plant complexes (2).

4.1.2. Connectivity analysis

In this section, we always assume that the complexes are non-empty, so we have $\underline{\xi} \preccurlyeq \underline{\delta}$ for plant complexes, and $\underline{\xi} \leq \underline{\delta}$ for colored plant complexes. For similar connectivity proofs, cf. [Wah13, Sect. 4] and [Tra14, Sect. 2].

Proposition 4.8. *Both* $|\,\mathrm{F}\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)|$ *and* $|\,\mathrm{F}\mathcal{O}^{\underline{\xi}}_{\underline{\delta}}(S)|$ *are contractible.*

Proof. The main tool of the proof is the construction of a flow similar to the *Hatcher flow* introduced in [Hat91]. The proof is carried out for $|\,\mathrm{F}\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)|$. It is fully analogous for $|\,\mathrm{F}\mathcal{O}^{\underline{\xi}}_{\underline{\delta}}(S)|$.

In the following, we switch freely between plants as subsets of S, plants as vertices of plant complexes, and their respective isotopy classes if no misunderstandings are possible.

We fix a plant v in $\mathrm{F}\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)$ with arcs $a_1, \ldots, a_\xi$ in a fixed order. Our goal is to show that $|\,\mathrm{F}\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)|$ deformation retracts onto $|\,\mathrm{Star}(v)|$, which is contractible by Lemma A.5.

We order the interior points of v in the following way:

- $x \prec y$ if $x \in a_i$, $y \in a_j$ for $i < j$
- If $x, y \in a_i$, $x \prec y$ if x is closer to $*$ along a_i than y.

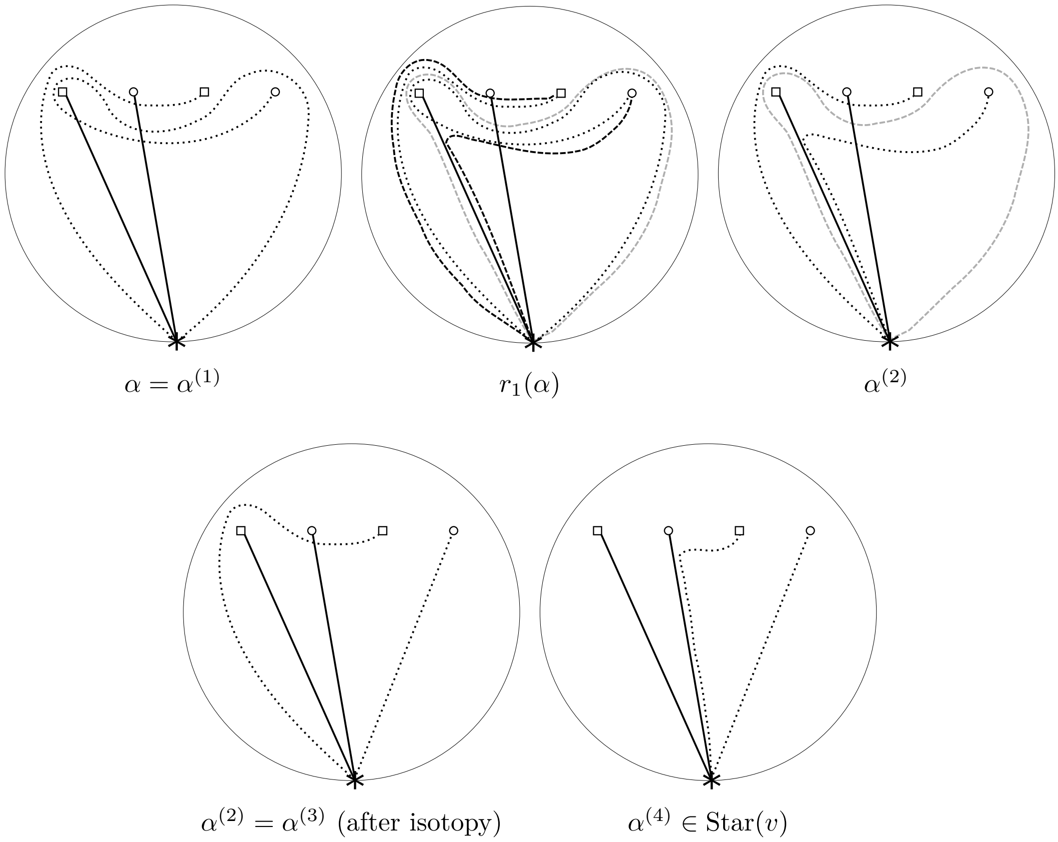


Figure 4.5.: Construction of the flow for a 0-simplex α: The solid arcs constitute v. Grey paths will be removed.

Let $\alpha = \langle w_0, \ldots, w_p \rangle$ be a p-simplex of $\mathrm{F}\mathcal{P}^{\xi}_{\underline{\delta}}(S)$ with representative plants w_i chosen such that the number of intersections with v is minimal. The ordering of the interior points of v induces an order on the set $(w_0 \cup \ldots \cup w_p) \cap v^\circ$ of intersection points, which we denote by $g_1, \ldots, g_k$. At g_i, the plant w_{j_i} intersects the arc a_{l_i}.

In an ϵ-neighborhood of g_1, we erase the segments of the arc of w_{j_1} which contains g_1 and join the two loose ends to $*$ by straight lines. We denote by $C(\alpha)$ the plant that is obtained from w_{j_1} by replacing the arc containing g_1 with a smooth approximation of the one of the two newly created paths that is an arc (the dashed black arc in the second picture of Figure 4.5, while the dashed grey loop remains unused). Because of the order we put on the intersection points, $C(\alpha)$ is a plant which is disjoint from the plants in $\alpha = \alpha^{(1)}$.

In the following, we misuse notation by allowing vertices to occur more than once in a simplex. In this sense, by $\langle c_0, \ldots, c_p \rangle$ we denote the simplex with vertices $\{c_0, \ldots, c_p\}$, which might be of dimension strictly smaller than p.

We now define a finite sequence of simplices inductively. We start with $i = 1$, the first intersection point $g_i = g_1$, and the simplex

$$r_1(\alpha) = \langle w_0, \ldots, w_p, C(\alpha)\rangle = \langle \alpha^{(1)}, C(\alpha^{(1)})\rangle$$

and execute the procedure below. In every step, we choose representative plants for the vertices of the simplices such that the number of intersections with v is minimal.

(1) Increase i by one. Stop if $i = k + 1$, otherwise go to the next step.

(2) If the intersection at g_i is not yet resolved in $\alpha^{(i)}$, replace the plant of $\alpha^{(i)}$ that contains g_i with $C(\alpha^{(i)})$, denote the resulting p-simplex by $\alpha^{(i+1)}$, and set $r_i(\alpha) = \langle \alpha^{(i)}, C(\alpha^{(i)})\rangle$. Else, set $\alpha^{(i+1)} = \alpha^{(i)}$ and $r_i(\alpha) = \langle \alpha^{(i)}, w'_{j_i}\rangle$, where w'_{j_i} is the j_i-th plant of $\alpha^{(i)}$.

(3) Go to step 1.

By the above remarks about disjointness, we produce simplices $r_i(\alpha)$ of dimension at most $p + 1$ at each step. The p-simplex $\alpha^{(k+1)}$ is in the star of v, since all of its plants are disjoint from v. By construction, $\alpha^{(k+1)}$ is a face of $r_k(\alpha)$.

Now, we may use the sequence $r_1(\alpha), \ldots, r_k(\alpha)$ to define a deformation retraction of $|\operatorname{F}\mathcal{P}^{\xi}_{\underline{\delta}}(S)|$ onto $|\operatorname{Star}(v)|$. Using barycentric coordinates[1], any point on the realization of the p-simplex $\alpha = \langle w_0, \ldots, w_p\rangle$ can be identified with a tuple $T = (t_0, \ldots, t_p)$, where $t_j \geq 0$ and $\sum_{j=0}^{p} t_j = 1$. For $j = 0, \ldots, p$, let $k_j = |w_j^\circ \cap v^\circ| = w_i.v$, where the second equality is due to the choice of the w_i. Given a tuple T and $i \in \{1, \ldots, k\}$, we assign to g_i the weight $\omega_i(T) = t_{j_i}/k_{j_i}$ if $k_{j_i} > 0$, and $\omega_j(T) = 0$ else, such that $\sum_{i=1}^{k} \omega_i(T) = 1$.

For fixed α and T, we define $f\colon I \to |\operatorname{F}\mathcal{P}^{\xi}_{\underline{\delta}}(S)|$ by

$$f^T_\alpha(s) = [r_i(\alpha), (x_0, \ldots, x_{p+1})]$$

for $\sum_{j=1}^{i-1} \omega_j(T) \leq s \leq \sum_{j=1}^{i} \omega_j(T)$, $i = 1, \ldots, k$. Here, we set $x_l = t_l$ for all l, except for the pair

$$(x_{j_i}, x_{p+1}) = \left(t_{j_i} - k_{j_i}\left(s - \sum_{j=1}^{i-1} \omega_j \right), k_{j_i}\left(s - \sum_{j=1}^{i-1} \omega_j \right)\right).$$

[1] Here, we use the same order on the vertices of $\operatorname{F}\mathcal{P}^{\xi}_{\underline{\delta}}(S)$ as in the definition of the $r_i(\alpha)$. If a vertex c_j appears more than once in $r_i(\alpha)$, adding up the corresponding entries of a given tuple T yields the barycentric coordinate of the point which we refer to.

The map f_α^T is well-defined as the definition of the $r_i(\alpha)$ fits to the intervals for s:

$$f_\alpha^T\left(\sum_{j=1}^{i}\omega_j\right) = [r_{i+1}(\alpha),(t_0,\dots,t_p,0)] = [r_i(\alpha),(t_0,\dots,t_{j_i-1},0,t_{j_i+1},\dots,t_p,t_{j_i})].$$

By construction, $f_\alpha^T(1)$ lies in $\alpha^{(k+1)} \in \mathrm{Star}(v)$. We may now patch the f_α^T for all simplices α and coordinates T with non-zero entries in order to obtain a global map $f\colon I \times |\,\mathrm{F}\mathcal{P}_{\underline{\delta}}^{\underline{\xi}}(S)| \to |\,\mathrm{F}\mathcal{P}_{\underline{\delta}}^{\underline{\xi}}(S)|$ with image in $\mathrm{Star}(v)$.

We still need to prove that f is continuous. By [Spa66, Thm. 3.1.15], we only have to show continuity for the restriction of f to the geometric realization of any simplex.

In the interior of the realization of $\alpha = \langle w_0,\dots,w_p\rangle$, continuity follows from the definition of the $\omega_i(T)$. It remains to show that we may go to a subsimplex of α continuously. That is, for $\beta = \langle w_0,\dots,w_{p-1}\rangle$, we must show that for all $s \in I$, $f_\alpha^{(t_0,\dots,t_{p-1},0)}(s) = f_\beta^{(t_0,\dots,t_{p-1})}(s)$.

In essence, the reason for this is Lemma 4.2: The number of intersections of w_p and v does not depend on the simplex α. In other words, we have $v.\beta = v.\alpha - v.w_p$. For $[\alpha,(t_0,\dots,t_{p-1},0)]$, all weights concerning intersections with w_p are zero, and all other weights concern intersection points also in $\beta \cap v^\circ$. Thus, the weights for intersection points not on w_p coincide for $[\alpha,(t_0,\dots,t_{p-1},0)]$ and $[\beta,(t_0,\dots,t_{p-1})]$. The ordering of the intersection points is not affected by removing w_p. Concluding, we have $\omega'_j(t_0,\dots,t_{p-1},0) = \omega_{q_j}(t_0,\dots,t_{p-1})$ for some indices q_j, and $q_i < q_j$ if $i < j$, where the $\omega'_j(t_0,\dots,t_{p-1},0)$ are the weights for the points in $\beta \cap v^\circ$.

Furthermore, the steps in the procedure at which $r_i(\beta)$ and $r_{q_i}(\alpha)$ are created are identical. The remaining steps (the $r_j(\alpha)$ where $j \neq q_i$ for any i, i.e., removing intersection points with w_p) are unimportant in this case since the corresponding weights for $[\alpha,(t_0,\dots,t_{p-1},0)]$ are zero. Thus, we can pass from α to β continuously. For the other facets of α, the proof is identical. For an arbitrary subsimplex, the claim follows inductively. $\square$

For a p-simplex α of $\mathcal{P}_{\underline{\delta}}^{\underline{\xi}}$, we write S_α for the connected space $(S \setminus \alpha) \cup \{*\}$. We can define (colored) plant complexes on S_α accordingly. In particular, the arguments from Proposition 4.8 carry over to $S = S_\alpha$, so spaces of the form $|\,\mathrm{F}\mathcal{P}_{\underline{\delta}}^{\underline{\xi}}(S_\alpha)|$ ($|\,\mathrm{F}\mathcal{O}_{\underline{\delta}}^{\underline{\xi}}(S_\alpha)|$) are contractible for $\underline{\xi} \preccurlyeq \underline{\delta}$ ($\underline{\xi} \leq \underline{\delta}$).

Proposition 4.9. *Let* $\min \underline{\xi} > 0$. *The space* $|\,\mathcal{P}_{\underline{\delta}}^{\underline{\xi}}(S)|$ *is at least* $\left(\left\lfloor \frac{\min \underline{\delta}}{2\max \underline{\xi}} \right\rfloor - 2\right)$*-connected.*

The following easy inequality will be useful in the proof of the proposition.

Lemma 4.10. *For all integers $a \geq b > 0$, the inequality $a - b \cdot \lfloor \frac{a}{2b} \rfloor \geq b$ is valid.*

Proof. For $a < 2b$, we have $a - b \cdot \lfloor \frac{a}{2b} \rfloor = a - b \cdot 0 = a \geq b$, while in the case $a \geq 2b$, $a - b \cdot \lfloor \frac{a}{2b} \rfloor \geq a - b \cdot \frac{a}{2b} = \frac{a}{2} \geq b$ holds. □

Proof of Proposition 4.9. We prove the proposition for a surface with boundary S or a space of the form S_γ, where γ is a collection of arcs in (S, Δ).

The claim is proved by induction on $\min \underline{\delta}$, with $\max \underline{\xi}$ fixed. For $\min \underline{\delta} < 2 \cdot \max \underline{\xi}$, the statement of the proposition is void. We assume from now on that we have $\max \underline{\xi} \leq \min \underline{\delta}$.

For the inductive step, let $k \leq \left\lfloor \frac{\min \underline{\delta}}{2 \max \underline{\xi}} \right\rfloor - 2$, and consider a map $f\colon S^k \to |\, \mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)|$. We have to show that f factors through a $(k+1)$-disk. By the contractibility of $|\, \mathrm{F}\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)|$, we have a commutative diagram:

$$\begin{tikzcd} S^k \arrow[r, "f"] \arrow[d, hook] & |\, \mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)| \arrow[d, hook] \\ D^{k+1} \arrow[r, "\hat{f}"] & |\, \mathrm{F}\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)| \end{tikzcd}$$

By Proposition A.6, we may assume that all maps are simplicial. That is, they are the geometric realization of simplicial maps $\mathcal{F}\colon \mathcal{S}^k \to \mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)$ and $\hat{\mathcal{F}}\colon \mathcal{D}^{k+1} \to \mathrm{F}\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)$ for some finite PL triangulations $\mathcal{S}^k$ and $\mathcal{D}^{k+1}$ of the k-sphere and the $(k+1)$-disk, respectively. Therefore, it suffices to show that $\mathcal{F}$ factors through $\mathcal{D}^{k+1}$.

Our goal is to deform $\hat{\mathcal{F}}$ such that its image lies entirely in $\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)$. We call a simplex α of $\mathcal{D}^{k+1}$ *bad* if in each plant in $\hat{\mathcal{F}}(\alpha)$, there is at least one arc that shares an endpoint with an arc from another plant in $\hat{\mathcal{F}}(\alpha)$ (note that vertices are good). In particular, a simplex of $\mathcal{D}^{k+1}$ with image in $\mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S)$ cannot contain any bad subsimplices.

Let α be a bad simplex of $\mathcal{D}^{k+1}$ of maximal dimension $p \leq k+1$ among all bad simplices. Now, $\hat{\mathcal{F}}$ restricts to a map

$$\hat{\mathcal{F}}|_{\mathrm{Link}(\alpha)}\colon\ \mathrm{Link}(\alpha) \to J_\alpha \coloneqq \mathcal{P}^{\underline{\xi}}_{\underline{\delta}'}(S_{\hat{\mathcal{F}}(\alpha)}),$$

where $\underline{\delta}'$ is obtained from $\underline{\delta}$ by removing the endpoints of the arcs in α from an instance of $\underline{\delta}$. We still need to argue why the image of $\mathrm{Link}(\alpha)$ lies in $\mathcal{P}^{\underline{\xi}}_{\underline{\delta}'}(S_{\hat{\mathcal{F}}(\alpha)})$: If it did not, there would be a bad simplex $\beta \in \mathrm{Link}(\alpha)$, hence $\alpha * \beta$ would be bad, contradicting the maximality of the dimension of α (note that α and β are joinable as β is in $\mathrm{Link}(\alpha)$).

Now, any p-simplex uses at most $(p+1)\cdot\max\underline{\xi}$ endpoints of any Δ_i, so we have

$$\min\underline{\delta}' \geq \min\underline{\delta} - (p+1)\cdot\max\underline{\xi}\,. \tag{4.5}$$

Furthermore, we have $p+1 \leq k+2 \leq \left\lfloor \frac{\min\underline{\delta}}{2\max\underline{\xi}} \right\rfloor$, so we obtain from (4.5), the assumption $\min\underline{\delta} \geq \max\underline{\xi}$, and Lemma 4.10:

$$\begin{aligned}
\min\underline{\delta}' &\geq \min\underline{\delta} - (p+1)\cdot\max\underline{\xi} \\
&\geq \min\underline{\delta} - \max\underline{\xi}\cdot\left\lfloor \frac{\min\underline{\delta}}{2\max\underline{\xi}} \right\rfloor \\
&\geq \max\underline{\xi}\,.
\end{aligned}$$

From the assumption $\min\underline{\xi} > 0$, we get $\min\underline{\delta}' < \min\underline{\delta}$. Therefore, the induction hypothesis is applicable to $J_\alpha = \mathcal{P}^{\underline{\xi}}_{\underline{\delta}'}(S_{\hat{\mathcal{F}}(\alpha)})$:

$$\begin{aligned}
\operatorname{conn} J_\alpha &\geq \left\lfloor \frac{\min\underline{\delta}'}{2\max\underline{\xi}} \right\rfloor - 2 \\
&\geq \left\lfloor \frac{\min\underline{\delta} - (p+1)\cdot\max\underline{\xi}}{2\max\underline{\xi}} \right\rfloor - 2 \\
&\geq \left\lfloor \frac{\min\underline{\delta}}{2\max\underline{\xi}} - p \right\rfloor - 2 \qquad (4.6) \\
&\geq k - p,
\end{aligned}$$

where (4.6) follows from $p \geq 1$.

The rest is **standard machinery**, cf. also the end of the proof of [Wah13, Thm. 4.3]: By this connectivity bound for J_α, as the link of α is a $(k+1)-p-1 = (k-p)$-sphere, there is a commutative diagram

$$\begin{tikzcd}
\operatorname{Link}(\alpha) \arrow[r, "\hat{\mathcal{F}}|_{\operatorname{Link}(\alpha)}"] \arrow[d, hook] & J_\alpha \arrow[r] & \mathcal{P}^{\underline{\xi}}_{\underline{\delta}}(S) \\
K \arrow[ur, "\hat{f}'"'] & &
\end{tikzcd}$$

with K a $(k-p+1)$-disk with boundary $\partial K = \operatorname{Link}(\alpha)$. The right map identifies plants on S_α with plants on S. Now, in the triangulation $\mathcal{D}^{k+1}$, replace the $(k+1)$-disk $\operatorname{Star}(\alpha) = \alpha * \operatorname{Link}(\alpha)$ with the $(k+1)$-disk $\partial\alpha * K$. This works because both $\operatorname{Star}(\alpha)$

and $\partial\alpha * K$ have the same boundary $\partial\alpha * \mathrm{Link}(\alpha)$. On $\partial\alpha * K$, modify $\hat{\mathcal{F}}$ by

$$\hat{\mathcal{F}} * \hat{\mathcal{F}}' \colon \partial\alpha * K \to \mathrm{F}\mathcal{P}_{\underline{\delta}}^{\underline{\xi}}(S).$$

This is possible since $\hat{\mathcal{F}}'$ agrees with $\hat{\mathcal{F}}$ on $\mathrm{Link}(\alpha) = \partial K$.

New simplices in $\partial\alpha * K$ are of the form $\tau = \beta_1 * \beta_2$, where β_1 is a proper face of α and β_2 is mapped to J_α. Therefore, if τ is a bad simplex in $\partial\alpha * K$, then $\tau = \beta_1$ since plants of $\hat{\mathcal{F}}'(\beta_2)$ do not share any endpoints with other plants of $\hat{\mathcal{F}}'(\beta_2)$ or $\hat{\mathcal{F}}'(\beta_1)$, so they cannot contribute to a bad simplex. But β_1 is a proper face of α, so we have decreased the number of top dimensional bad simplices. By induction on the number of top dimensional bad simplices, the result follows. $\square$

The connectivity bound given in Proposition 4.9 is useless for (m,r)-ferns, where $\min\underline{\delta} = \max\underline{\xi} = r$. In the following proposition, we generalize a result from [Tra14], where the $(1,r)$-fern complex is proved to be $(t-2)$-connected, where t is the number of entries in $\underline{\delta}$.

Proposition 4.11. *Let $r > 0$, $t \geq m > 0$, and let $\mathcal{P}_{\underline{\delta}}^{\underline{\xi}}(S)$ be the (m,r)-fern-complex on S. Then, we have*

$$\operatorname{conn}|\,\mathcal{P}_{\underline{\delta}}^{\underline{\xi}}(S)| \geq \left\lfloor \frac{t}{2m-1} \right\rfloor - 2.$$

Proof. The proof is similar to the proof of Proposition 4.9 and is performed by induction on t, fixing m and r. The multifern complex $\mathcal{P}_{\underline{\delta}}^{\underline{\xi}}(S)$ is always non-empty, so the base case $t = m$ is trivial, as are all cases with $t < 4m - 2$. We assume from now on that $t \geq 2m$ holds.

Let now $k \leq \left\lfloor \frac{t}{2m-1} \right\rfloor - 2$ and argue as in Proposition 4.9: We want to extend a simplicial map $\mathcal{F} \colon \mathcal{S}^k \to \mathcal{P}_{\underline{\delta}}^{\underline{\xi}}(S)$ to a triangulation of the disk $\mathcal{D}^{k+1}$, using the same notion of a *bad* simplex as before. Let α be a bad simplex of $\mathcal{D}^{k+1}$ of maximal dimension p, so $\hat{\mathcal{F}} \colon \mathcal{D}^{k+1} \to \mathrm{F}\mathcal{P}_{\underline{\delta}}^{\underline{\xi}}(S)$ restricts to

$$\hat{\mathcal{F}}|_{\mathrm{Link}(\alpha)} \colon \mathrm{Link}(\alpha) \to J_\alpha := \mathcal{P}_{\underline{\delta}'}^{\underline{\xi}}(S_{\hat{\mathcal{F}}(\alpha)}),$$

where $\underline{\delta}'$ is the remainder of $\underline{\delta}$ after removing the arcs of α. Let t' be the number of positive entries of $\underline{\delta}'$. An arbitrary p-simplex uses at most $(p+1)m$ different Δ_i. As α is bad, it uses at most $(p+1)(m-1) + \lfloor \frac{p+1}{2} \rfloor$ different Δ_i. Recall that by the definition of multiferns, a multifern β having endpoints at Δ_i necessarily implies that

Δ_i disappears in S_β. That is, we have

$$\begin{aligned}
t' &\geq t - \left((p+1)(m-1) + \left\lfloor \frac{p+1}{2} \right\rfloor\right) && (4.7)\\
&= t - \left\lfloor (p+1)\left(m - \frac{1}{2}\right) \right\rfloor \\
&\geq t - \left\lfloor \left\lfloor \frac{t}{2m-1} \right\rfloor \left(m - \frac{1}{2}\right) \right\rfloor && (4.8)\\
&\geq t - \left\lfloor \frac{t}{2} \right\rfloor \\
&\geq m. && (4.9)
\end{aligned}$$

Here, (4.8) is due to the fact that $p+1 \leq k+2 \leq \left\lfloor \frac{t}{2m-1} \right\rfloor$, and (4.9) is true since we demanded that $t \geq 2m$. Consequently, we can apply the induction hypothesis to J_α. Using (4.7), we obtain

$$\begin{aligned}
\operatorname{conn}(J_\alpha) &\geq \left\lfloor \frac{t'}{2m-1} \right\rfloor - 2 \\
&\geq \left\lfloor \frac{t - (p+1)(m-1) - \left\lfloor \frac{p+1}{2} \right\rfloor}{2m-1} - 2 \right\rfloor \\
&\geq \left\lfloor k - \frac{\left\lfloor (p+1)\left(m - \frac{1}{2}\right) \right\rfloor}{2m-1} \right\rfloor \\
&\geq \left\lfloor k - \frac{p+1}{2} \right\rfloor \\
&\geq k - p,
\end{aligned}$$

as $p \geq 1$ (there are no bad vertices).

The rest of the proof can be copied from Proposition 4.9 (*standard machinery*). □

Remark 4.12. The *arc complex* $\mathcal{A}(S,t)$ can be defined as a plant complex in two ways: It is the complex $\mathcal{A}(S,t) = \mathcal{P}_{\underline{\delta}}^{\underline{\xi}}(S)$ for either $\underline{\delta} = (t)$ and $\underline{\xi} = (1)$ (*cofern construction*) or for $\underline{\delta}$ arbitrary and $\underline{\xi} = (1, 0, \ldots, 0)$ (*fern construction*).

Now, we obtain a lower connectivity bound of $\lfloor \frac{n}{2} \rfloor - 2$ from the cofern construction and Proposition 4.9, and a bound of $n-2$ from the fern construction (e.g., $\underline{\delta} = (1, \ldots, 1)$) and Proposition 4.11. The latter bound is the same as the one given in [HW10, Thm. 7.2]. For $S = D$ a closed disk, the arc complex is in fact contractible, cf. [Dam13].

We now turn our attention to the connectivity of *colored* plant complexes:

Proposition 4.13. *Let* $\min \underline{\xi} > 0$. *We have* $\operatorname{conn} |\mathcal{O}^{\underline{\xi}}_{\underline{\delta}}(S)| \geq \min_{i=1,\ldots,t} \left\lfloor \frac{\delta_i}{2\xi_i} \right\rfloor - 2$.

Proof. The proof is basically analogous to the one of Proposition 4.9. The claim is void if there is an $i \in \{1, \ldots, t\}$ such that $\delta_i < 2\xi_i$, and we assume for the proof that $\delta_i \geq \xi_i$ for all i. Choose a bad simplex α of maximal dimension $p \leq k+1 \leq \min_{i=1,\ldots,t} \left\lfloor \frac{\delta_i}{2\xi_i} \right\rfloor - 1$. In analogy to (4.5), we obtain

$$\begin{aligned} \delta_i' &\geq \delta_i - (p+1) \cdot \xi_i \\ &\geq \delta_i - \xi_i \cdot \min_{j=1,\ldots,t} \left\lfloor \frac{\delta_j}{2\xi_j} \right\rfloor \\ &\geq \delta_i - \xi_i \left\lfloor \frac{\delta_i}{2\xi_i} \right\rfloor \\ &\geq \xi_i \end{aligned}$$

for all $i = 1, \ldots, t$. For the image J_α of $\operatorname{Link}(\alpha)$ in $\mathrm{F}\mathcal{O}^{\underline{\xi}}_{\underline{\delta}}(S)$, we now have

$$\begin{aligned} \operatorname{conn} J_\alpha &\geq \min_{i=1,\ldots,t} \left\lfloor \frac{\delta_i'}{\xi_i} \right\rfloor - 2 \\ &\geq \min_{i=1,\ldots,t} \left\lfloor \frac{\delta_i - (p+1) \cdot \xi_i}{2\xi_i} - 2 \right\rfloor \\ &= \left\lfloor \min_{i=1,\ldots,t} \left(\frac{\delta_i}{2\xi_i} \right) - 2 - \frac{p+1}{2} \right\rfloor \\ &\geq k - p, \end{aligned}$$

since $p \geq 1$. The remainder of the proof, again, is *standard machinery* from the proof of Proposition 4.9. $\square$

4.2. Combinatorics of colored plant complexes

In this section, we focus on a specific class of colored plant complexes on a disk D which provides a crucial tool for the deduction of homological stability results for Hurwitz spaces in Chapter 5. Let $\underline{\xi}$ be a non-increasing t-tuple of positive integers and $n \in \mathbb{N}$. We consider colored plant complexes of the form

$$\mathcal{O}^{[n,\underline{\xi}]} := \mathcal{O}^{\underline{\xi}}_{n \cdot \underline{\xi}}(D)$$

and write $\mathcal{O}^{[n,\underline{\xi}]}_q$ for the set of q-simplices of $\mathcal{O}^{[n,\underline{\xi}]}$. Because of the specific constellation $\underline{\delta} = n \cdot \underline{\xi}$, it is clear that the dimension of $\mathcal{O}^{[n,\underline{\xi}]}$ equals $n - 1$.

The braid action We now identify a group action on these complexes which makes a combinatorial description of $\mathcal{O}^{[n,\underline{\xi}]}$ possible. See Section 2.2.2 for the definition of the groups $\mathrm{Br}_{n\cdot\underline{\xi}} \subset \mathrm{Br}_{n\xi}$.

Lemma 4.14. *For all $0 \leq q < n$, the group $\mathrm{Br}_{n\cdot\underline{\xi}}$ acts on $\mathcal{O}_q^{[n,\underline{\xi}]}$.*

Proof. By Proposition 2.9, we can identify the full braid group $\mathrm{Br}_{n\xi}$ with the mapping class group $\mathrm{Map}(D_{n\xi})$ of a disk with $n\xi$ marked points, where the standard generator σ_i corresponds to a half twist in counterclockwise direction which interchanges the i-th and the $(i+1)$-th marked point, cf. Section 2.1.1. If we partition the $n\xi$ marked points into t disjoint sets of respective cardinalities $n\xi_1, \ldots, n\xi_t$, the subgroup $\mathrm{Br}_{n\cdot\underline{\xi}} \subset \mathrm{Br}_{n\xi}$ can be identified with the set of mapping classes which leave this partition invariant.

Thus, $\mathrm{Br}_{n\cdot\underline{\xi}}$ acts on D by isotopy classes of diffeomorphisms. This gives a well-defined action on $\mathcal{O}^{[n,\underline{\xi}]}$ since the vertices of the complex are isotopy classes of colored plants. By bijectivity, the non-intersecting property is preserved. In particular, the set of q-simplices in $\mathcal{O}^{[n,\underline{\xi}]}$ is invariant under the $\mathrm{Br}_{n\cdot\underline{\xi}}$-action. □

For the action of $\sigma \in \mathrm{Br}_{n\cdot\underline{\xi}}$ on a simplex $\alpha \in \mathcal{O}^{[n,\underline{\xi}]}$, we write $\alpha \mapsto \sigma \cdot \alpha$.

Let $\alpha \in \mathcal{O}_q^{[n,\underline{\xi}]}$ be a q-simplex. The sorting of the inward pointing unit tangent vectors of representative arcs of α at $*$ in clockwise order is well-defined, cf. Remark 4.7. As the mapping classes in $\mathrm{Br}_{n\cdot\underline{\xi}}$ fix the boundary pointwise and are orientation-preserving, this ordering is invariant under the $\mathrm{Br}_{n\cdot\underline{\xi}}$-action.

Using this ordering, we assign an *index* $i \in \{0, \ldots, q\}$ and a *color* $j \in \{1, \ldots, t\}$ to each arc in α:

- An arc is labeled with the index i if it belongs to the $(i+1)$-th plant in α, where we use the order on the plants of α described in Remark 4.7.
- An arc is labeled with the color j if its endpoint lies in Δ_j.

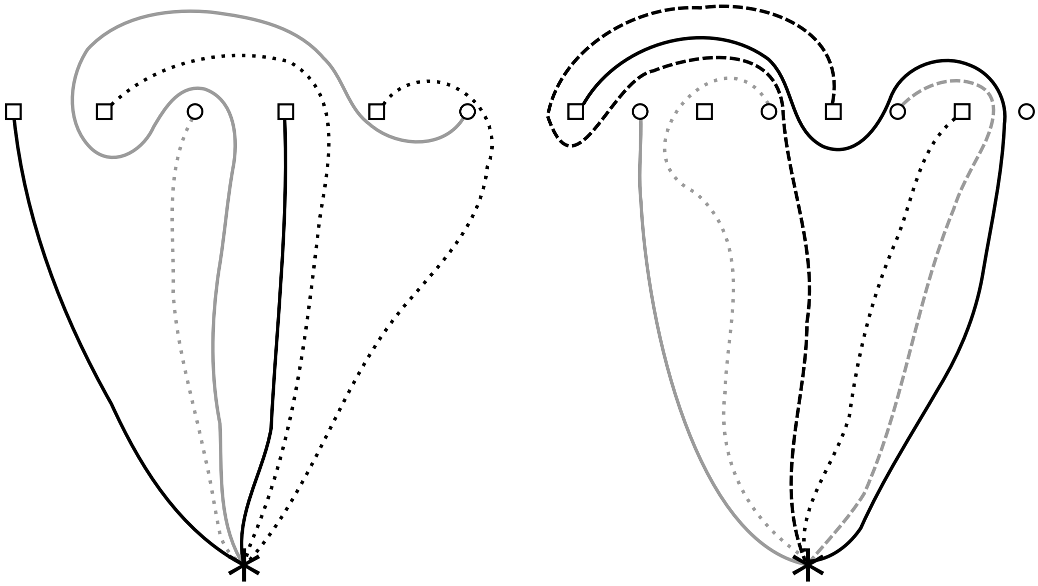
Consequently, any q-simplex α defines a unique sequence

$$\omega_\alpha = (i_1, j_1), (i_2, j_2), \ldots, (i_{(q+1)\cdot\xi}, j_{(q+1)\cdot\xi}), \tag{4.10}$$

where i_k is the index and j_k the color of the k-th arc of α, in clockwise order at $*$. By the above reasoning, this sequence is $\mathrm{Br}_{n\cdot\underline{\xi}}$-invariant.

Definition 4.15. We call (4.10) the *IC-sequence* (index-color-sequence) of $\alpha \in \mathcal{O}_q^{[n,\underline{\xi}]}$.

With the help of IC-sequences, we are able to count the orbits of the braid action from Lemma 4.14:



$\omega_\alpha = (0,1),(1,2),(0,2),(0,1),(1,1),(1,1) \qquad \omega_\beta = (0,2),(1,2),(2,1),(1,1),(2,2),(0,1)$

Figure 4.6.: IC-sequences of simplices $\alpha \in \mathcal{O}_1^{[2,(2,1)]}$ and $\beta \in \mathcal{O}_2^{[4,(1,1)]}$.

Lemma 4.16. *Let $q < n$. The set $\mathcal{O}_q^{[n,\underline{\xi}]}$ decomposes into*

$$l_q^{\underline{\xi}} := \left| \mathcal{O}_q^{[n,\underline{\xi}]} \,/\, \mathrm{Br}_{n\cdot\underline{\xi}} \right| = \frac{(\xi(q+1))!}{(q+1)! \cdot (\xi_1! \cdot \ldots \cdot \xi_t!)^{q+1}}$$

$\mathrm{Br}_{n\cdot\underline{\xi}}$*-orbits. The set of orbits in $\mathcal{O}_q^{[n,\underline{\xi}]}$ corresponds to the set of occurring IC-sequences for q-simplices.*

Remark 4.17. A priori, the number $l_q^{\underline{\xi}}$ depends not only on q and $\underline{\xi}$ but also on n. As a consequence of the lemma, the actual quantity is independent on n as long as $q < n$; therefore, it makes sense to omit n from the notation.

Proof. We show that the $\mathrm{Br}_{n\cdot\underline{\xi}}$-orbits are in bijection to the $l_q^{\underline{\xi}}$ possible IC-sequences for q-simplices.

We start by counting the number of possible IC-sequences. We say a sequence $(i_1, j_1), (i_2, j_2), \ldots, (i_{(q+1)\cdot\xi}, j_{(q+1)\cdot\xi})$ is *q-feasible* if each index $i \in \{0, \ldots, q\}$ is assigned ξ times; for each index, each color $j = \{1, \ldots, t\}$ is assigned ξ_j times; and the index $i+1$ does not appear before the index i, for all $i = 0, \ldots, q-1$.

Since D is path-connected, a q-feasible sequence appears as the IC-sequence of a q-simplex if and only if $q < n$, i.e., as long as there are enough endpoints for the arcs.

We may now count the number of q-feasible sequences: There are

$$\binom{\xi(q+1)}{\underbrace{\xi,\dots,\xi}_{(q+1)\text{ times}}}\frac{1}{(q+1)!}=\frac{(\xi(q+1))!}{(q+1)!\cdot(\xi!)^{q+1}}$$

different partitionings of $\xi(q+1)$ arcs into subsets of size ξ, where we used a multinomial coefficient. Any such partition gives a unique indexing (recall that the first arc with index i appears before the first arc with index $i+1$).

Given an index i, the ξ arcs labeled with it can be colored in $\binom{\xi}{\xi_1,\dots,\xi_t}$ different ways, which makes it

$$l_q^{\underline{\xi}}=\frac{(\xi(q+1))!}{(q+1)!\cdot(\xi!)^{q+1}}\cdot\binom{\xi}{\xi_1,\dots,\xi_t}^{q+1}=\frac{(\xi(q+1))!}{(q+1)!\cdot(\xi_1!\cdot\dots\cdot\xi_t!)^{q+1}}$$

different choices of q-feasible sequences, as there are $(q+1)$ different indices.

We have already seen above that the IC-sequence of a q-simplex is invariant under the $\mathrm{Br}_{n\cdot\underline{\xi}}$-action. In the second part of the proof, we will now show that the group $\mathrm{Br}_{n\cdot\underline{\xi}}$ acts transitively on simplices with according IC-sequences by an argument similar to the proof of transitivity in [EVW16, Prop. 5.6]. An alternate proof can be obtained by adapting the methods from the proof of [Wah13, Prop. 2.2(1)].

Let α,β be two q-simplices with the same IC-sequence ω. We choose representative non-intersecting collections of the plants of α and β with arcs $a_1,\dots,a_{(q+1)\xi}$ and $b_1,\dots,b_{(q+1)\xi}$, respectively, subscripts chosen such that the arcs are arranged in clockwise order at $*$. Since $\mathrm{Br}_{n\cdot\underline{\xi}}$ surjects onto $\mathcal{S}_{n\cdot\underline{\xi}}$, we may assume that a_i and b_i have the same endpoint $a_i(1)=b_i(1)$ for all $i=1,\dots,(q+1)\xi$. Furthermore, after a suitable isotopy, we may as well assume that for some $\epsilon>0$, we have $a_i(t)=b_i(t)$ for all $0\le t\le\epsilon$. Hence, if we choose a continuous increasing function $h\colon I\to\mathbb{R}$ with $h(t)=t$ for $0\le t\le\epsilon/2$ and $h(1)=\epsilon$, we obtain $a_i\circ h=b_i\circ h$ for all i.

It remains to show that there is an orientation-preserving homeomorphism G of D which, for all $i=1,\dots,(q+1)\xi$, retracts the arc a_i to $a_i\circ h$, and fixes those marked points which are no endpoints of arcs in α. In addition, we construct a similar map H which carries b_i to $b_i\circ h$ for all i. Then, the homeomorphism $H^{-1}\circ G$ defines a mapping class which carries α to β, and which corresponds to an element in $\mathrm{Br}_{n\cdot\underline{\xi}}$ because the IC-sequences of α and β coincide. To construct G, choose disjoint closed tubular neighborhoods U_i of $a_i|_{[\epsilon/3,1]}$ for all i. Such neighborhoods exist since the arcs are disjoint except at $*$. Now, U_i is homeomorphic to a closed disk, and so there exists a homeomorphism (which can in fact be realized as an isotopy of the identity) which restricts to the identity on ∂U_i and which carries the arc segment $U_i \cap a_i$ to its retraction $U_i \cap (a_i \circ h)$. Combining these homeomorphisms and extending them by the identity on $D \setminus \bigcup_i U_i$ yields the desired homeomorphism G. The construction of H is analogous. $\square$

Lemma 4.18. *Let $q < n$. For any simplex $\alpha \in \mathcal{O}_q^{[n,\underline{\xi}]}$, the stabilizer of α under the $\mathrm{Br}_{n\cdot\underline{\xi}}$-action is isomorphic to $\mathrm{Br}_{(n-q-1)\cdot\underline{\xi}}$. In particular, there is a bijection between the elements of the orbit $\mathrm{Br}_{n\cdot\underline{\xi}} \cdot\alpha$ and the cosets in $\mathrm{Br}_{n\cdot\underline{\xi}} / \mathrm{Br}_{(n-q-1)\cdot\underline{\xi}}$.*

Proof. We show that the stabilizer of a simplex $\alpha \in \mathcal{O}_q^{[n,\underline{\xi}]}$ is isomorphic to $\mathrm{Br}_{(n-q-1)\cdot\underline{\xi}}$. Then, the second assertion follows directly from the orbit-stabilizer-theorem.

Let $\Sigma \subset D$ be the union of a representative set of the arcs of α, only intersecting at $*$. In clockwise order around $*$, denote the arcs in Σ by $a_1, \ldots, a_{(q+1)\xi}$. We denote by $\mathrm{Map}(D_{(n-q-1)\cdot\underline{\xi}}, \Sigma)$ the group of isotopy classes of orientation-preserving diffeomorphisms of $D_{(n-q-1)\cdot\underline{\xi}}$ which fix Σ pointwise. Since Σ is contractible, $\mathrm{Map}(D_{(n-q-1)\cdot\underline{\xi}}, \Sigma)$ is isomorphic to the group $\mathrm{Map}(D_{(n-q-1)\cdot\underline{\xi}}) \cong \mathrm{Br}_{(n-q-1)\cdot\underline{\xi}}$, with marked points given by the marked points in $D_{n\cdot\underline{\xi}} \setminus \Sigma$. We will show that the inclusion of subgroups of $\mathrm{Map}(D_{n\cdot\underline{\xi}})$

$$\mathrm{Map}\left(D_{(n-q-1)\cdot\underline{\xi}}, \Sigma\right) \hookrightarrow \left(\mathrm{Map}\left(D_{n\cdot\underline{\xi}}\right)\right)_\alpha$$

is surjective and hence an isomorphism.

For this part, we follow the similar proofs of [Wah13, Prop. 2.2(2)] and [EVW16, Prop. 5.6]. Choose an element $\phi \in \mathrm{Diff}^+(D_{n\cdot\underline{\xi}})$ which stabilizes the simplex α. We have to show that ϕ is isotopic to a diffeomorphism that fixes Σ pointwise.

By definition, $\phi(a_1)$ is isotopic to a_1. The isotopy extension theorem (cf. [Pal60]) implies that we can extend a corresponding isotopy to an ambient isotopy, so we may assume that ϕ fixes a_1 pointwise. We proceed by induction on the number of fixed arcs. Let $j > 1$, and assume that ϕ fixes $\Sigma_j = a_1 \cup \ldots \cup a_{j-1}$ pointwise. The arc a_j is isotopic to $\phi(a_j)$, and we must show that the corresponding isotopy can be chosen disjointly from Σ_j. If this holds, another application of the isotopy extension theorem implies the inductive step and thus the statement.

Let $H \colon I \times I \to D$ be a smooth isotopy that conveys $\phi(a_j)$ to a_j, and assume that H is transverse to Σ_j, using the transversality theorem ([Tho54]). Here, $H(0, -)$ and $H(1, -)$ correspond to the arcs $\phi(a_j)$ and a_j, respectively. Furthermore, we have $H(-, 0) = *$, and $H(-, 1) \in \Delta$ is the endpoint of a_j.

Now, consider the preimage $H^{-1}(\Sigma_j)$. The line $I \times \{0\}$ is the preimage of $*$, and by transversality, all other components must be circles in the interior of $I \times I$.

Since the intersection number is finite, there is at least one such circle which encloses no further circle in $H^{-1}(\Sigma_j)$. Let D_0 be the closed disk it encloses. Let furthermore $\Sigma_j^\delta \subset D$ be a closed δ-thickening of Σ_j with $\delta > 0$ chosen such that Σ_j^δ is still contractible. By continuity of H, we may now choose $\epsilon > 0$ such that for a closed ϵ-neighborhood D_0^ϵ of D_0, we have $H(\partial D_0^\epsilon) \subset \Sigma_j^\delta$,

Restriction of H to the closed disk D_0^ϵ defines an element of $\pi_2(D, \Sigma_j^\delta \setminus \Sigma_j)$. This relative homotopy group is trivial by the long exact sequence of relative homotopy groups (cf. [Hat02, Thm. 4.3]), since D and $\Sigma_j^\delta \setminus \Sigma_j$ are contractible onto $*$. We may thus replace H on D_0^ϵ by a homotopic map H' with $H'|_{\partial D_0^\epsilon} = H|_{\partial D_0^\epsilon}$ and image in $\Sigma_j^\delta \setminus \Sigma_j$, which exists since $\Sigma_j^\delta \setminus \Sigma_j$ is simply connected.

By extending H' to $I \times I$ by $H'|_{(I\times I)\setminus D_0^\epsilon} = H|_{(I\times I)\setminus D_0^\epsilon}$, we obtain a homotopy H' with $\pi_0(H'^{-1}(\Sigma_j)) < \pi_0(H^{-1}(\Sigma_j))$. Proceeding inductively, we construct a homotopy H'' which is disjoint from Σ_j. Finally, by [Eps66, Thm. 3.1], H'' can be replaced by an isotopy in $(D \setminus \Sigma_j) \cup \{*\}$. $\square$

Standard simplices From the definition, it is clear that any ordered simplicial complex has the structure of a semi-simplicial set, cf. Definition A.8.

As a consequence of Lemma 4.16 and Lemma 4.18, there are bijections between the set $\mathcal{O}_q^{[n,\underline{\xi}]}$ of q-simplices and the disjoint union of $l_q^{\underline{\xi}}$ copies of $\mathrm{Br}_{n\cdot\underline{\xi}} / \mathrm{Br}_{(n-q-1)\cdot\underline{\xi}}$ for all $q = 0, \ldots, n-1$. Our next goal is to make these bijections compatible with the semi-simplicial structure on $\mathcal{O}^{[n,\underline{\xi}]}$: We want to fix bijections and describe the structure of a semi-simplicial set on

$$\mathbf{O}^{[n,\underline{\xi}]} = \coprod_{q=0}^{n-1} \mathbf{O}_q^{[n,\underline{\xi}]} = \coprod_{q=0}^{n-1} \left(\coprod_{l_q^{\underline{\xi}} \text{ copies}} \mathrm{Br}_{n\cdot\underline{\xi}} / \mathrm{Br}_{(n-q-1)\cdot\underline{\xi}} \right)$$

which is compatible with the face maps, hence defines a *semi-simplicial isomorphism* $\mathbf{O}^{[n,\underline{\xi}]} \cong \mathcal{O}^{[n,\underline{\xi}]}$.

We fix $n, \underline{\xi}$, and q. After applying a suitable homeomorphism, we may assume that D lies in the complex plane as a disk of radius 1 centered at $0 \in \mathbb{C}$, that we have $* = -\mathrm{i}$, and that the $n\xi$ marked points in D are all real and arranged from left to right in n *clusters* of ξ points each, where ξ_i points in each cluster lie in Δ_i, for $i = 1, \ldots, t$. For each of these clusters, we suppose that the points in Δ_i are placed to the left of the points in Δ_j, for $1 \le i < j \le t$.

Let now ω be a fixed IC-sequence. In what follows, we define a standard q-simplex in $\mathcal{O}_q^{[n,\underline{\xi}]}$ for the IC-sequence ω algorithmically: We resort the terms of ω by the con-

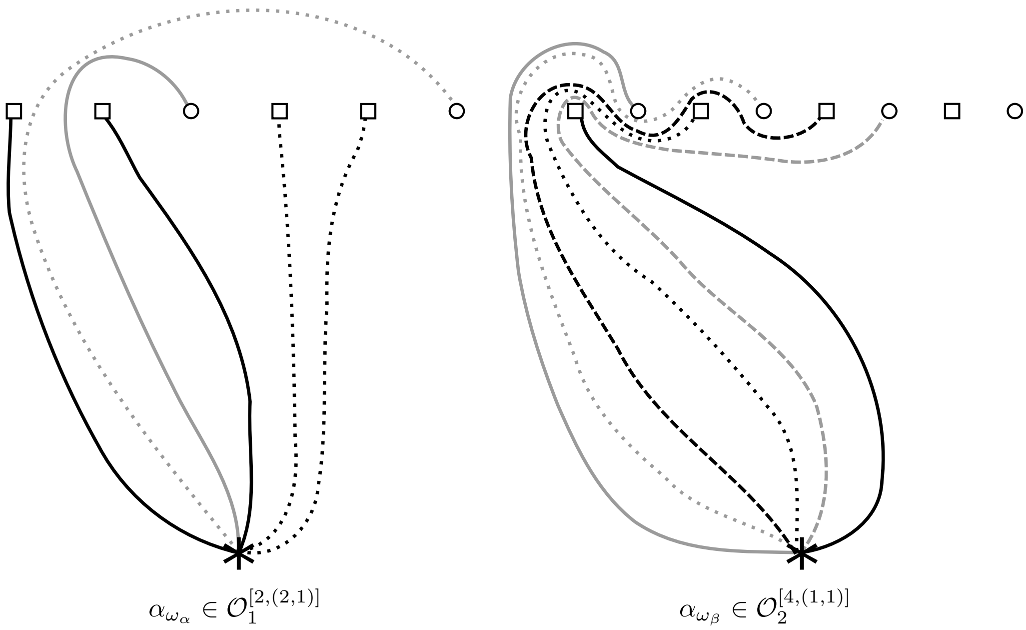


Figure 4.7.: Standard simplices for the IC-sequences of α and β from Figure 4.6.

secutive sorting criteria *index* (1st), *color* (2nd) and *position in the IC-sequence* (3rd), and draw the arcs of a q-simplex in this new order, respecting the order at $*$ prescribed by the IC-sequence. We draw the arcs in such a fashion that the endpoint of each arc is chosen as the leftmost free marked point, where we always *undercross* marked points if possible. This is compatible with the coloring because of the arrangement of marked points.

This process generates a set of $(q+1) \cdot \xi$ arcs which are disjoint away from $*$ and unique up to isotopy since we work on a disk D. Distinguishing by the indices, these arcs can be divided into $q+1$ colored $\underline{\xi}$-plants, which in turn define a q-simplex $\alpha_\omega \in \mathcal{O}_q^{[n,\underline{\xi}]}$. Using these simplices, every q-simplex can be written as $\sigma \cdot \alpha_\omega$ for some IC-sequence ω and some $\sigma \in \mathrm{Br}_{n\cdot\underline{\xi}}$. This is a consequence of the transitivity of the $\mathrm{Br}_{n\cdot\underline{\xi}}$-action on simplices with the same IC-sequence, cf. Lemma 4.16.

Definition 4.19. The simplex $\alpha_\omega \in \mathcal{O}_q^{[n,\underline{\xi}]}$ is called the *standard simplex for the IC-sequence ω.*

Definition 4.20. Let $\alpha \in \mathcal{O}_q^{[n,\underline{\xi}]}$ be a q-simplex, and ω its IC-sequence. The sequence $\widetilde{\omega} = (p_1, p_2, \ldots, p_{(q+1)\cdot\xi})$ induced by the reordering of the IC-sequence of α described above, where p_i is the position in the IC-sequence of the corresponding arc, is called the *P-sequence* (position sequence) of α.

Example 4.21. The P-sequences of the simplices depicted in Figures 4.6 and 4.7 are given by $\tilde{\omega}_\alpha = (1, 4, 3, 5, 6, 2)$ and $\tilde{\omega}_\beta = (6, 1, 4, 2, 3, 5)$.

Remark 4.22. By definition, for $i = 0, \dots, q$, the positions from $p_{i\xi+1}$ through $p_{(i+1)\cdot\xi}$ correspond to arcs of the same colored plant.

Furthermore, we see that the set of P-sequences is in natural bijection to the set of IC-sequences. We use the different notations for the $\mathrm{Br}_{n\cdot\underline{\xi}}$-orbits in $\mathcal{O}_q^{[n,\underline{\xi}]}$ depending on their usefulness in the respective situations.

We identify $\mathrm{Br}_{n\xi}$ with the mapping class group of the $n\xi$-punctured disk in such a way that $\mathrm{Br}_{n\cdot\underline{\xi}}$ stabilizes of the colored configuration of $n\xi$ points in D described above. Hence, the element $\sigma_{i\xi+j}$, for $i = 0, \dots, q$ and $j = 1, \dots, \xi - 1$, describes the isotopy class of a a half twist that interchanges the j-th and the $(j + 1)$-th point of the $(i + 1)$-th cluster. On the other hand, the elements of the form $\sigma_{i\xi}$, $i = 1, \dots, q$ describe a half twist that interchanges the ξ-th point of the i-th cluster with the first point of the $(i + 1)$-th cluster.

We know from Lemma 4.18 that the stabilizer of a q-simplex is isomorphic to the group $\mathrm{Br}_{(n-q-1)\cdot\underline{\xi}}$. For any standard q-simplex α_ω, we may thus write

$$(\mathrm{Br}_{n\cdot\underline{\xi}})_{\alpha_\omega} = \langle \sigma_k \mid (q + 1) \cdot \xi + 1 \leq k \leq n\xi - 1 \rangle \cap \mathrm{Br}_{n\cdot\underline{\xi}} \,.$$

As this expression is independent of ω, we may denote the stabilizer of *any* standard q-simplex by

$$L_q = (\mathrm{Br}_{n\cdot\underline{\xi}})_{\alpha_\omega} \cong \mathrm{Br}_{(n-q-1)\cdot\underline{\xi}} \,.$$

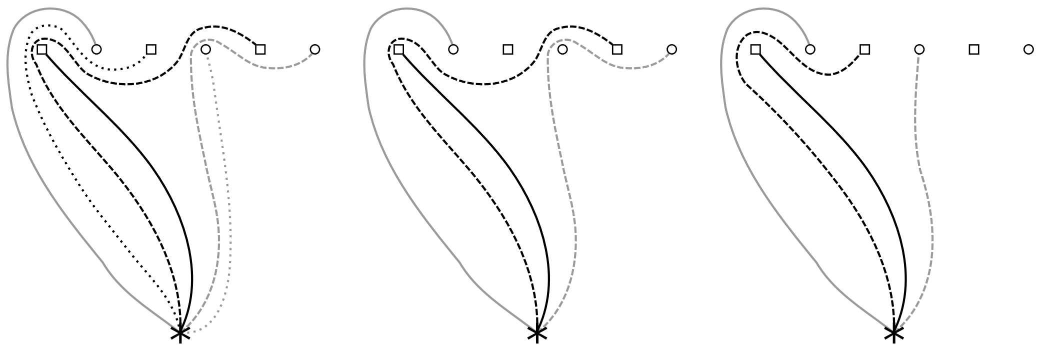
Now, once and for all, we fix the bijection

$$\begin{aligned} \Gamma_\omega \colon \mathrm{Br}_{n\cdot\underline{\xi}} /L_q &\to \mathrm{Br}_{n\cdot\underline{\xi}} \cdot\alpha_\omega \\ \sigma L_q &\mapsto \sigma \cdot \alpha_\omega \end{aligned}$$

for each $\mathrm{Br}_{n\cdot\underline{\xi}}$-orbit in $\mathcal{O}^{[n,\underline{\xi}]}$. Collecting these maps for all IC-sequences, we obtain a global bijection

$$\begin{aligned} \Gamma \colon \mathbf{O}^{[n,\underline{\xi}]} &\to \mathcal{O}^{[n,\underline{\xi}]} \\ (\omega_p, \sigma L_p) &\mapsto \sigma \cdot \alpha_{\omega_p} \quad \text{for all } p \geq 0, \end{aligned}$$

where ω_p is an IC-sequence of a p-simplex.



α_ω $\partial_1\alpha_\omega$ $\alpha_{d_1\omega}$

Figure 4.8.: The simplices α_ω, $\partial_1\alpha_\omega$, $\alpha_{d_1\omega} \in \mathcal{O}^{[3,(1,1)]}$ for ω from Example 4.23.

Face maps Recall that the colored plants in a q-simplex $\alpha = \langle v_0, \dots, v_q\rangle \in \mathcal{O}_q^{[n,\underline{\xi}]}$ are ordered by the tangential direction at $*$ of their respective leftmost arcs. For $i = 0, \dots, q$, the i-th *face map* is given by leaving out the vertex v_i:

$$\begin{aligned} \partial_i \colon \mathcal{O}_q^{[n,\underline{\xi}]} &\to \mathcal{O}_{q\text{-}1}^{[n,\underline{\xi}]} \\ \langle v_0, \dots, v_q\rangle &\mapsto \langle v_0, \dots, \hat{v}_i, \dots v_q\rangle. \end{aligned}$$

We now determine *face maps* in $\mathbf{O}_q^{[n,\underline{\xi}]}$ which are compatible with the face maps in $\mathcal{O}^{[n,\underline{\xi}]}$ insofar as they give $\mathbf{O}^{[n,\underline{\xi}]}$ the structure of a semi-simplicial set isomorphic to $\mathcal{O}^{[n,\underline{\xi}]}$. These maps are evidently given by $\Gamma^{-1} \circ \partial_i \circ \Gamma$. For use in later chapters, we need to describe them explicitly.

Given a q-simplex with IC-sequence ω, its i-th face map is given by deleting the arcs with index i. Hence, we may define the i-th *face* of ω as the IC-sequence $d_i\omega$ obtained by first removing all the pairs with index i from ω, and secondly subtracting 1 from the indices of the remaining elements with indices bigger than i. The IC-sequence $d_i\omega$ defines a P-sequence which we denote by $d_i\tilde{\omega}$.

Example 4.23. Consider the IC-sequence $\omega = (0,2),(1,1),(2,1),(0,1),(2,2),(1,2)$ for a simplex in $\mathcal{O}^{[3,(1,1)]}$. The corresponding P-sequence is $\tilde{\omega} = (4,1,2,6,3,6)$, and the faces d_1 of the sequences are given by $d_1\omega = (0,2),(1,1),(0,1),(1,2)$ and $d_1\tilde{\omega} = (3,1,2,4)$. The inherent standard simplices are depicted in Figure 4.8.

Our next goal is to find elements $\tau_{i,q}^{\omega} \in \mathrm{Br}_{n\cdot\underline{\xi}}$ for all $i = 0, \dots, q$, such that

(i) $\partial_i\alpha_\omega = \tau_{i,q}^{\omega} \cdot \alpha_{d_i\omega}$, and

(ii) $\tau_{i,q}^{\omega}$ commutes with L_{q-1}.

If we identify such elements, we are eventually able to define maps

$$\begin{aligned} \mathbf{d}_i \colon\ \mathbf{O}_q^{[n,\underline{\xi}]} &\to \mathbf{O}_{q\text{-}1}^{[n,\underline{\xi}]} \\ (\omega, \sigma L_q) &\mapsto (d_i\omega, \sigma\tau_{i,q}^{\omega} L_{q-1}), \end{aligned}$$

which are independent of the choice of a representative for the coset σL_q because of condition (ii) and the fact that we have $L_q \subset L_{q-1}$. Furthermore, by condition (i), such elements satisfy

$$\begin{aligned} \mathbf{d}_i(\omega, \sigma L_q) &= (d_i\omega, \sigma\tau_{i,q}^{\omega} L_{q-1}) \\ &= \Gamma^{-1}(\sigma\tau_{i,q}^{\omega} \cdot \alpha_{d_i\omega}) \\ &= \Gamma^{-1}(\sigma \cdot \partial_i\alpha_\omega) \\ &= (\Gamma^{-1} \circ \partial_i)(\sigma \cdot \alpha_\omega) \qquad (4.11) \\ &= (\Gamma^{-1} \circ \partial_i \circ \Gamma)(\omega, \sigma L_q), \end{aligned}$$

as desired. Here, in (4.11), we used the geometric fact that for any $\sigma \in \mathrm{Br}_{n\cdot\underline{\xi}}$, we have $\sigma \cdot \partial_i\alpha_\omega = \partial_i(\sigma \cdot \alpha_\omega)$: The plant deletion operator ∂_i commutes with the action of the mapping class defined by σ.

The face of an IC-sequence of a simplex is defined as the IC-sequence of the corresponding face of a simplex. By Lemma 4.16, the colored braid group $\mathrm{Br}_{n\cdot\underline{\xi}}$ acts transitively on the set of simplices with the same IC-sequence. From these two facts, it is immediate that elements $\tau_{i,q}^{\omega}$ satisfying condition (i) exist and that the coset $\tau_{i,q}^{\omega} L_{q-1}$ is unique. We are yet to determine them explicitly, and check whether they satisfy condition (ii).

By the construction of standard simplices, the first i plants of $\partial_i\alpha_\omega$ and $\alpha_{d_i\omega}$ are identical. Now, mapping $\alpha_{d_i\omega}$ to $\partial_i\alpha_\omega$ requires transferring the points of the $(q+1)$-th cluster to the $(i+1)$-th cluster in a suitable way: This transfer is performed for one point after the other, starting with the leftmost point. Let $\widetilde{\omega} = (p_1, \ldots, p_{(q+1)\cdot\xi})$ be the P-sequence of α_ω. If $p_{i\xi+m} < p_{r\xi+s}$ for some $m, s \in \{1, \ldots, \xi\}$ and $q \geq r > i$, the m-th point of the $(q+1)$-th cluster has to be half-twisted around the endpoint of the arc which (in $\alpha_{d_i\omega}$) ends at the s-th point of the r-th cluster in a positive direction, and in a negative direction otherwise. A careful analysis of this procedure (where we take into account that we let the braid group act from the left) yields the following formula:

$$\tau_{i,q}^{\omega} = \prod_{j=1}^{\xi} \left(\prod_{k=(i+1)\cdot\xi}^{(q+1)\cdot\xi-1} (\sigma_{k-j+1})^{\mathrm{sgn}(p_{k+1} - p_{(i+1)\cdot\xi-j+1})} \right) \qquad (4.12)$$

Here, $\mathrm{sgn}\colon \mathbb{Z} \to \{-1, 1\}$ is the signum function. Visibly, the largest index of a braid generator involved is $(q+1)\cdot\xi - 1$, so $\tau^{\omega}_{q,i}$ commutes with the elements of L_q, where the smallest index involved is $(q+1)\cdot\xi + 1$. Thus, condition (ii) is satisfied.

An example for the stepwise construction of $\tau^{\omega}_{i,1}$ can be found in Figure 4.9.

Remark 4.24.

(i) A priori, $\tau^{\omega}_{i,q} \in \mathrm{Br}_{n\cdot\underline{\xi}}$ for some fixed $n > q$. We note that the definition in (4.12) does not depend on the particular choice of n, so we can regard $\tau^{\omega}_{i,q}$ as a common element of all $\mathrm{Br}_{n\cdot\underline{\xi}}$ for $n > q$, using the inclusions $\mathrm{Br}_{n\cdot\underline{\xi}} \hookrightarrow \mathrm{Br}_{(n+1)\cdot\underline{\xi}}$ given by attaching ξ trivial strands with coloring $\underline{\xi}$ to the right of a braid in $\mathrm{Br}_{n\cdot\underline{\xi}}$.

(ii) If $\tilde{\omega}$ is the P-sequence corresponding to the IC-sequence ω, we sometimes also write $\tau^{\tilde{\omega}}_{i,q}$ to denote the element $\tau^{\omega}_{i,q}$.

We have just finished the proof of the following result:

Proposition 4.25. *For $i = 0, \dots, q$, the maps*

$$\begin{aligned} \mathbf{d}_i\colon\ \mathbf{O}^{[n,\underline{\xi}]}_{q} &\to \mathbf{O}^{[n,\underline{\xi}]}_{q\text{-}1} \\ (\omega, \beta L_q) &\mapsto (d_i\omega, \beta\tau^{\omega}_{i,q}L_{q-1}), \end{aligned}$$

give $\mathbf{O}^{[n,\underline{\xi}]}$ the structure of a semi-simplicial set such that $\Gamma\colon \mathbf{O}^{[n,\underline{\xi}]} \to \mathcal{O}^{[n,\underline{\xi}]}$ is an isomorphism of semi-simplicial sets.

Remark 4.26. As there is a $\mathrm{Br}_{n\cdot\underline{\xi}}$-action on $\mathcal{O}^{[n,\underline{\xi}]}$ by Lemma 4.14, there is also a $\mathrm{Br}_{n\cdot\underline{\xi}}$-action on $\mathbf{O}^{[n,\underline{\xi}]}$: A braid $\tau \in \mathrm{Br}_{n\cdot\underline{\xi}}$ acts via $\tau\cdot(\omega_p, \sigma L_p) = (\omega_p, \tau\sigma L_p)$.

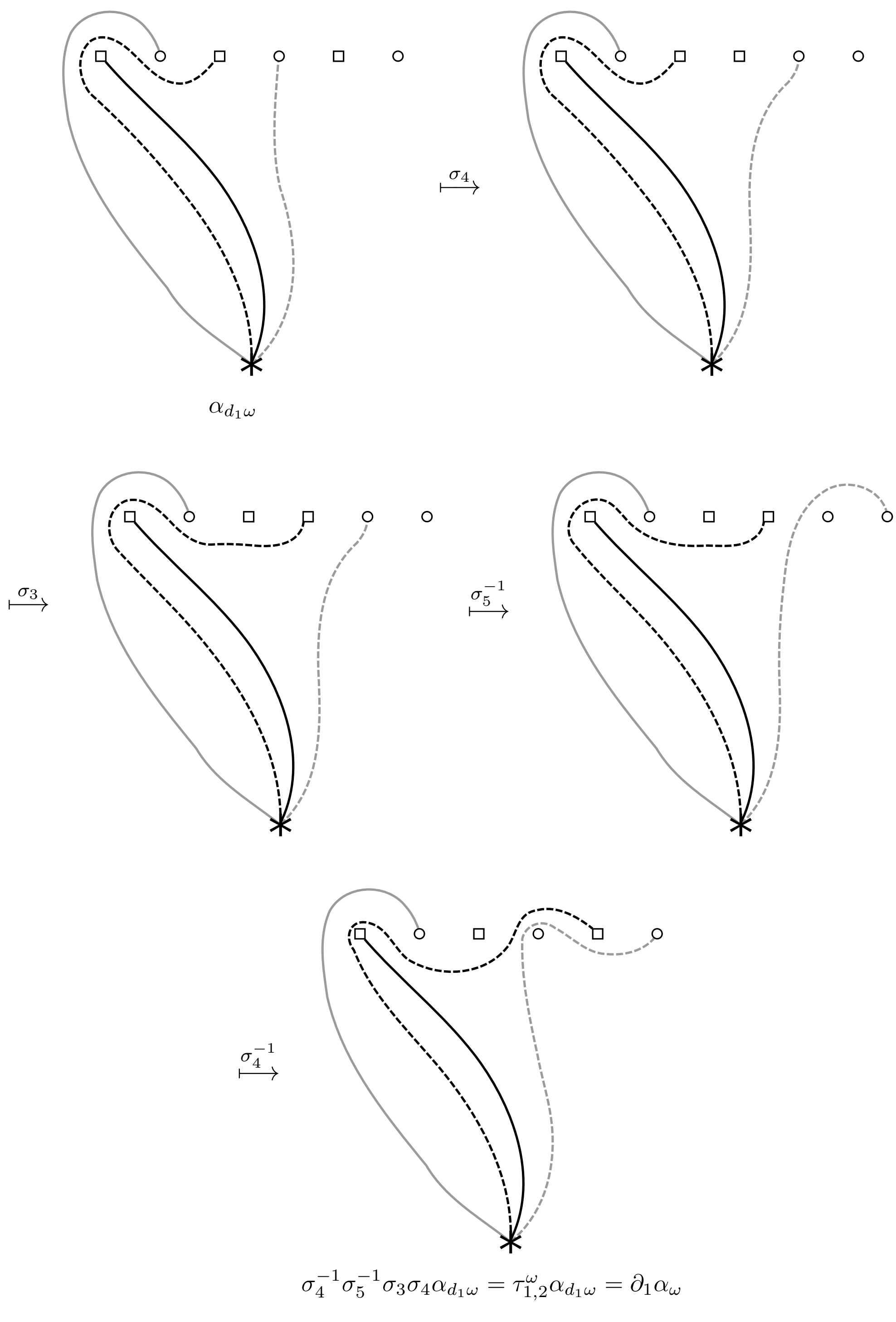


Figure 4.9.: Passing from $\alpha_{d_1\omega}$ to $\partial_1\alpha_\omega$ (ω from Example 4.23).

5. Homological Stability for Hurwitz Spaces

In this chapter, we prove Theorem 5.7 which is our central result in its most general form. It provides a general criterion for the homological stabilization of the sequence of Hurwitz spaces $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ defined in Section 3.2.

Our proof follows the ideas of Sections 4 through 6 of [EVW16], the first paper about homological stability for Hurwitz spaces by Ellenberg, Venkatesh, and Westerland. The highly connected *colored plant complexes* from Section 4.1 and the associated combinatorial properties presented in Section 4.2 are the essential new tool for the extension of the prior results. These allow us to define a spectral sequence converging to the homology of Hurwitz spaces. The main technical complication in comparison to the prior result is the fact that the colored braid group action on the set of q-simplices of the colored plant complexes is in general not transitive.

In what follows, we use the notion of the *ring of connected components* introduced in Definition 3.14. Let G be a finite group, $c = (c_1, \ldots, c_t)$ a tuple of distinct conjugacy classes in G, and $\underline{\xi} \in \mathbb{N}^t$. Let moreover A be a commutative ring. We usually write R instead of $R^{A,c}_{G,\underline{\xi}}$. For a central element $U \in R$ and $R[U]$ the U-torsion in R, we use the notation $D_R(U) = \max\{\deg(R/UR), \deg(R[U])\}$.

5.1. Abelian covers

Before investigating the more complicated homological setup for the general case in this chapter's main part, we start with the case of *Abelian covers*. We make use of the following (non-standard) definitions:

Definition 5.1. Let G be a finite group. A marked G-cover of D branched at $S \subset D^\circ$ with monodromy $\mu\colon \pi_1(D \setminus S) \to G$ is called

(i) *Abelian* if the image of μ is an abelian group, and

(ii) *purely Abelian* if the image of μ is a subgroup of the center of G.

We say that $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ *parametrizes (purely) Abelian covers* if all covers in $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ are (purely) Abelian.

Note that a connected cover is Abelian if and only if G is abelian (note the different choice of cases for the word *Abelian/abelian* depending on the context). Thus, in the connected case, our definition of Abelian covers agrees with the standard notion.

Example 5.2. If G is an abelian group, $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ parametrizes purely Abelian covers in any case. On the other hand, if $G = \mathcal{S}_3$ and $c = c_3$ is the conjugacy class of 3-cycles, $\mathrm{Hur}^c_{G,n}$ parametrizes Abelian covers which are *not* purely Abelian.

Remark 5.3. Let $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ parametrize Abelian covers.

(i) Let $X \subset \mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ be a connected component parametrizing covers with global monodromy $H \subsetneq G$. The points of X are disconnected Abelian covers, described by a branch locus S and an epimorphism $\mu\colon \pi_1(D\backslash S, *) \to H$. Hence, we may also interpret X as a connected component of $\mathrm{CHur}_{H,n\xi}$ with points corresponding to purely Abelian covers.

(ii) The Hurwitz action on $\mathbf{c}^n$ reduces to the permutation of entries, and each component of $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ is homotopy equivalent to a suitable colored configuration space. Hence, the number of $\mathrm{Br}_{n\cdot\underline{\xi}}$-orbits in $\mathbf{c}^n$ is given by the number of multisets of cardinality $n\xi$ that contain $n\xi_i$ elements from c_i, for all $i = 1, \ldots, t$:

$$b_0(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}) = \prod_{i=1}^{t} \binom{|c_i| + n\xi_i - 1}{n\xi_i}. \tag{5.1}$$

If we increase n, this number is constant if and only if all conjugacy classes in c are singletons, which is exactly the case of purely Abelian covers. Therefore, these are the only Hurwitz spaces parametrizing Abelian covers which give hope for homological stabilization.

(iii) By the two remarks above, we may state: If $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ parametrizes Abelian covers, its connected components are homotopic to suitable colored configuration spaces. If the covers are purely Abelian, $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ is as a whole homotopic to a colored configuration space.

From now on, we assume that $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ parametrizes purely Abelian covers. In this case, $\mathbf{c}^n = \left(c_1^{\xi_1} \times \ldots \times c_t^{\xi_t}\right)^n$ consists of a single Hurwitz vector with trivial $\mathrm{Br}_{n\cdot\underline{\xi}}$-action, so we obtain

$$\mathrm{Hur}^c_{G,n\cdot\underline{\xi}} = E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} \mathbf{c}^n = B\mathrm{Br}_{n\cdot\underline{\xi}} \simeq \mathrm{Conf}_{n\cdot\underline{\xi}}\,. \tag{5.2}$$

Let us now consider the ring R of connected components. As there is just a single Hurwitz vector in any degree, we obtain $R \cong A[x]$, where the indeterminate x corresponds to the unique element of $\mathbf{c}$. If we choose $U = x$, we clearly have $D_R(U) = 0$. In fact, a converse statement holds as well:

Lemma 5.4. *Let $U \in R_{>0}$ be a central homogeneous element. Then $D_R(U) = 0$ is possible if and only if $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ parametrizes purely Abelian covers.*

Proof. It remains to check the "only if"-part. If $\deg(R/UR) = \deg(R[U]) = 0$, the element U is necessarily of degree one and induces an isomorphism $R_i \cong R_{i+1}$ in any degree $i \geq 0$, so $R \cong A[x]$ must hold.

Now, assume that there is a conjugacy class c_1 in c which contains at least two elements $y \neq z$. Then, for two tuples $g = (y, g_2, \ldots, g_\xi)$, $h = (z, g_2, \ldots, g_\xi)$ in $\mathbf{c}$, we have $\partial g \neq \partial h$, so g and h cannot lie in the same $\mathrm{Br}_{\underline{\xi}}$-orbit. This contradicts $R_1 \cong A$. $\square$

If we combine (5.2), Theorem 2.23, and Lemma 5.4, we obtain immediately:

Corollary 5.5. *If there exists a central homogeneous element $U \in R_{>0}$ such that $D_R(U) = 0$ (equivalently, if $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ parametrizes purely Abelian covers), there is an isomorphism $H_p(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; \mathbb{Z}) \cong H_p(\mathrm{Hur}^c_{G,(n+1)\cdot\underline{\xi}}; \mathbb{Z})$ for $n \geq \frac{2p}{\min \underline{\xi}}$.*

Remark 5.6. Our separate treatment of the case $D_R(U) = 0$ is caused by estimates in the proofs of Lemma 5.33 and Theorem 5.7 which require $D_R(U) > 0$. In fact, the proof of an analogue of Proposition 5.34 can be simplified drastically in the purely Abelian case. Using this simplification, a stable range of $n \geq 2p + 2$ for Hurwitz spaces parametrizing purely Abelian covers can be achieved from the methods presented later in this chapter. As this does not improve on Tran's stable range given in Theorem 2.23, we omit this line of argument, making a case analysis in Sections 5.4 and 5.5 unnecessary.

5.2. The main theorem

Having solved the purely Abelian case, we may investigate the more diverse case where we assume $D_R(U) \geq 1$. Note that this happens for elements U of larger degree for purely Abelian covers as well, though this does not give us results improving on Corollary 5.5. Our main theorem is proved in Section 5.5:

Theorem 5.7. *Suppose there is a central homogeneous element $U \in R$ of positive degree such that $D_R = D_R(U)$ is positive and finite. Then, for any $p \geq 0$, multiplication by U induces an isomorphism*

$$H_p(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; A) \xrightarrow{\sim} H_p(\mathrm{Hur}^c_{G,(n+\deg U)\cdot\underline{\xi}}; A)$$

whenever $n > (8D_R + \deg U)p + 7D_R + \deg U$.

With the definition of the ring of connected components in mind, Theorem 5.7 indicates that homological stability for Hurwitz spaces depends solely on algebraic properties of their zeroth homology modules. In Chapter 6, we have a closer look at specific settings and give conditions for the existence of a homological stability theorem which are easier to handle. Beyond, in Section 6.2, we are able to determine the stable homology in a special (rather general) case, and pass from $\deg U$- to 1-periodic stability maps.

As a corollary of Theorem 5.7, Theorem 5.35 provides a criterion for homological stability for spaces of *unmarked* covers. In essence, we need to make sure that $U \in R$ is fixed under the conjugation action of G.

Theorem 5.7 is both a generalization of, and an improvement to, the prior result by Ellenberg, Venkatesh, and Westerland which deals with the case of a single *non-splitting* conjugacy class. Their motivation to study homological stability for Hurwitz spaces comes from arithmetics: The topological theorem is used to study the *Cohen-Lenstra heuristics*, cf. [EVW16, Sect. 1 & 8].

Definition 5.8. A conjugacy class $c \subset G$ is called *non-splitting* if c generates G and for any subgroup $H \subset G$, $c \cap H$ is either empty or a conjugacy class of H.

Theorem 5.9 (Ellenberg–Venkatesh–Westerland, [EVW16])
Let G be a finite group, $c \subset G$ a single non-splitting conjugacy class, and A a field whose characteristic is either zero or prime to $|G|$. Then there is a central homogeneous element $U \in R^{K,c}_{G,(1)}$ of positive degree and positive constants a, b (depending on G and c) such that U induces an isomorphism $H_p(\mathrm{Hur}^c_{G,n}; A) \xrightarrow{\sim} H_p(\mathrm{Hur}^c_{G,n+\deg U}; A)$ whenever $n > ap + b$.

In the present thesis, Theorem 5.7 extends Theorem 5.9 to the case of not necessarily constant Nielsen classes by giving conditions for the existence of a homological stability theorem for the corresponding Hurwitz spaces.

Remark 5.10. For their arithmetic application of Theorem 5.9, the authors of [EVW16] just need the existence of a ($\deg U$-periodic) homological stability theorem for Hurwitz spaces. The stable range which follows from the line of argument of their article is given by

$$n > (18D_R(U) + 3\deg U)p + 13D_R(U) + 3\deg U.$$

The stable range from Theorem 5.7 improves on this, which is mainly due to the fact that the authors of the prior paper did not attempt to optimize it.

The fact that Theorem 5.9 follows from Theorem 5.7 is a direct consequence of the following proposition. It states that the element U defined by Ellenberg, Venkatesh, and Westerland satisfies the algebraic conditions posed in Theorem 5.7:

Proposition 5.11 ([EVW16, Lemma 3.5])
Let G be a finite group, $c \subset G$ a non-splitting conjugacy class, and A a field whose characteristic is either zero or prime to $|G|$. Then, for some $T \gg 0$, the central homogeneous element

$$U = \sum_{g \in c} r(g)^{\mathrm{ord}(g)\cdot T} \in R^{c,K}_{G,(1)}$$

satisfies $D_R(U) < \infty$.

5.3. The spectral sequence

We are concerned with the spaces $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}} = E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} \mathbf{c}^n$, where $\mathbf{c} = c_1^{\xi_1} \times \ldots \times c_t^{\xi_t}$ as in Section 3.2.2. By rearranging the c_i if necessary, we may assume that the shape $\underline{\xi} = (\xi_1, \ldots, \xi_t)$ is non-increasing. We work with colored plant complexes of the form $\mathcal{O}^{[n,\underline{\xi}]} = \mathcal{O}^{\underline{\xi}}_{n\cdot\underline{\xi}}(D)$. As the construction of the spectral sequence in Proposition 5.14 only hinges on the semi-simplicial structure of $\mathcal{O}^{[n,\underline{\xi}]}$, we may as well work with the isomorphic semi-simplicial set $\mathbf{O}^{[n,\underline{\xi}]}$ by Proposition 4.25. In what follows, we thus switch freely between the geometric and the combinatorial construction of the simplicial complex without further mention.

Lemma 5.12. *The space $E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} (\mathcal{O}^{[n,\underline{\xi}]} \times \mathbf{c}^n)$ inherits the structure of a semi-simplicial set from the face maps ∂_i on $\mathcal{O}^{[n,\underline{\xi}]}$, where the left action of $\mathrm{Br}_{n\cdot\underline{\xi}}$ on the product $\mathcal{O}^{[n,\underline{\xi}]} \times \mathbf{c}^n$ is the diagonal action. The set of q-simplices is given by $E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} (\mathcal{O}^{[n,\underline{\xi}]}_q \times \mathbf{c}^n)$, with face maps $\tilde{\partial}_i[(e, \alpha, \underline{g})] = [(e, \partial_i\alpha, \underline{g})]$ for $i = 0, \ldots, q$.*

Proof. Since the face maps ∂_i on $\mathcal{O}^{[n,\underline{\xi}]}_q$ commute with the $\mathrm{Br}_{n\cdot\underline{\xi}}$-action, the $\tilde{\partial}_i$ are well-defined. Now for $i < j$, the semi-simplicial identity $\tilde{\partial}_i \circ \tilde{\partial}_j = \tilde{\partial}_{j-1} \circ \tilde{\partial}_i$ follows from the analogous statement for the ∂_i. □

Remark 5.13.

(i) The topology on the realization of the semi-simplicial set from Lemma 5.12 is the coarsest topology such that the canonical map

$$\psi\colon |\, E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} (\mathcal{O}^{[n,\underline{\xi}]} \times \mathbf{c}^n)| \to E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} (|\mathcal{O}^{[n,\underline{\xi}]}| \times \mathbf{c}^n)$$

is a homeomorphism. It maps $\varphi\colon E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} (\mathcal{O}_0^{[n,\underline{\xi}]} \times \mathbf{c}^n) \to I$, supported on a q-simplex $(e, \alpha, \underline{g})$ with vertices $(e, v_0, \underline{g}), \ldots, (e, v_q, \underline{g})$, to the triple

$$(e, \varphi'(e, \underline{g}), \underline{g}) \in E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} (|\mathcal{O}^{[n,\underline{\xi}]}| \times \mathbf{c}^n),$$

where the map $\varphi'(e, \underline{g})\colon \mathcal{O}_0^{[n,\underline{\xi}]} \to I$ is supported on $\alpha \in \mathcal{O}_q^{[n,\underline{\xi}]}$ and satisfies $\varphi'(e, \underline{g})(v_i) = \varphi(e, v_i, \underline{g})$.

(ii) The semi-simplicial structure on a simplicial complex $\mathcal{O}$ yields the structure of a semi-simplicial space on the geometric realization $|\mathcal{O}|$ with set of q-simplices given by

$$|\mathcal{O}|_q = \left\{ \varphi\colon \mathcal{O}_0 \to I \;\middle|\; \operatorname{supp}\varphi \in \mathcal{O}_q, \sum_{v \in \mathcal{O}_0} \varphi(v) = 1 \right\}.$$

We see that $|\mathcal{O}|_q$ is a disjoint union of open simplices. Up to homotopy, we may thus identify $|\mathcal{O}|_q$ with the discrete set $\mathcal{O}_q$.

Therefore, we use $E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} (\mathcal{O}_q^{[n,\underline{\xi}]} \times \mathbf{c}^n)$ as a homotopy model for the set of q-simplices of $E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} (|\mathcal{O}^{[n,\underline{\xi}]}| \times \mathbf{c}^n)$.

We obtain the following result:

Proposition 5.14. *There exists a homological spectral sequence E^1_{qp} which converges to $H_{p+q}(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; A)$ in degrees $p + q \leq \lfloor \frac{n}{2} \rfloor - 2$.*

Proof. By Proposition 4.13, the space $|\mathcal{O}^{[n,\underline{\xi}]}|$ is at least $\left(\lfloor \frac{n}{2} \rfloor - 2\right)$-connected. Now, the spectral sequence for the semi-simplicial space given by the geometric realization of the semi-simplicial set from Lemma 5.12 converges to the homology of the total space, cf. (B.1) in the Appendix and Remark 5.13:

$$H_p(E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} (\mathcal{O}_q^{[n,\underline{\xi}]} \times \mathbf{c}^n); A) = E^1_{qp} \Longrightarrow H_{p+q}(E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} (|\mathcal{O}^{[n,\underline{\xi}]}| \times \mathbf{c}^n); A).$$

By the Hurewicz theorem (cf. [Hat02, Thm. 4.32]) and applying the connectivity bound for $|\mathcal{O}^{[n,\underline{\xi}]}|$, we have $H_{p+q}(|\mathcal{O}^{[n,\underline{\xi}]}| \times \mathbf{c}^n; A) \cong H_{p+q}(\mathbf{c}^n; A)$ for $p + q \leq \lfloor \frac{n}{2} \rfloor - 2$

(recall that $\mathbf{c}^n$ is finite). Finally, passing to $\mathrm{Br}_{n\cdot\underline{\xi}}$-equivariant homology (cf. [Bro82, Ch. VII.7]), we obtain an isomorphism

$$\begin{aligned} H_{p+q}(E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} (|\mathcal{O}^{[n,\underline{\xi}]}| \times \mathbf{c}^n); A) &\cong H_{p+q}(E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} \mathbf{c}^n; A) \\ &= H_{p+q}(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; A) \end{aligned}$$

in the same range. □

Consider the spectral sequence from Proposition 5.14. For $q \geq n$, the set $\mathcal{O}_q^{[n,\underline{\xi}]}$ is empty, so $E^1_{qp} = 0$. Let now $q < n$. By employing the isomorphim $\mathcal{O}^{[n,\underline{\xi}]} \cong \mathbf{O}^{[n,\underline{\xi}]}$, we identify each of the $l_q^{\underline{\xi}}$ $\mathrm{Br}_{n\cdot\underline{\xi}}$-orbits in $\mathcal{O}_q^{[n,\underline{\xi}]}$ (cf. Lemma 4.16) with a copy of the quotient $\mathrm{Br}_{n\cdot\underline{\xi}}/L_q$. The subgroup $L_q \cong \mathrm{Br}_{(n-q-1)\cdot\underline{\xi}}$ acts on the last $(n-q-1)\cdot\xi$ entries of $\mathbf{c}^n$. Consequently, we obtain

$$\begin{aligned} E^1_{qp} &\cong H_p\left(\left\{1,\ldots,l_q^{\underline{\xi}}\right\} \times \left(E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} \left(\mathrm{Br}_{n\cdot\underline{\xi}}/L_q \times \mathbf{c}^n\right)\right)\right) \\ &\cong H_p\left(\left\{1,\ldots,l_q^{\underline{\xi}}\right\} \times \mathbf{c}^{q+1} \times \left(E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{L_q} \mathbf{c}^{n-q-1}\right); A\right) \qquad (5.3) \\ &\cong A^{l_q^{\underline{\xi}}} \otimes_A A\langle \mathbf{c}^{q+1}\rangle \otimes_A H_p(\mathrm{Hur}^c_{(n-q-1)\cdot\underline{\xi}}; A) =: \bar{E}^1_{qp} \qquad (5.4) \end{aligned}$$

where the first and second isomorphisms follow from the above remarks. The third isomorphism is due to the Eilenberg-Zilber theorem (cf. [Spa66, Thm. 5.3.6]) and the fact that $E\mathrm{Br}_{n\cdot\underline{\xi}}$ is a model for EL_q.

The differentials on the E^1-page are given by the alternating sum

$$\mathrm{d} = \sum_{i=0}^{q} (-1)^i \tilde{\partial}_{i*} \colon E^1_{qp} \to E^1_{q-1,p}.$$

Our next goal is to determine the corresponding maps $\bar{\mathrm{d}} \colon \bar{E}^1_{qp} \to \bar{E}^1_{q-1,p}$ induced by the isomorphisms $E^1_{qp} \cong \bar{E}^1_{qp}$ for any fixed p, where we set $\bar{E}^1_{qp} = 0$ for $q \geq n$.

Recall that by Remark 3.15, the ring R is generated as an A-algebra by degree one elements $r(s)$, with $s \in \mathbf{c}$. Now, we have a multiplication map

$$H_p(\mathrm{Hur}^c_{G,(n-q-1)\cdot\underline{\xi}}; A) \xrightarrow{r(s)\cdot} H_p(\mathrm{Hur}^c_{G,(n-q)\cdot\underline{\xi}}; A)$$

for any $s \in \mathbf{c}$, induced by the map (3.5).

The notation for tuples used below is introduced at the end of Section 1.3.

Lemma 5.15. *Let $q < n$. Under the isomorphism (5.4),* $\mathrm{d}\colon E^1_{qp} \to E^1_{q-1,p}$ *is represented by the linear map*

$$\bar{\mathrm{d}} = \sum_{i=0}^{q} (-1)^i \overline{\tilde{\partial}_{i*}}\colon \bar{E}^1_{qp} \to \bar{E}^1_{q-1,p},$$

the $\overline{\tilde{\partial}_{i*}}$*, for* $i = 0, \dots, q$*, being given by linear extension of*

$$\overline{\tilde{\partial}_{i*}}(\omega \otimes \underline{h} \otimes x) = d_i\omega \otimes \left(\left(\tau^{\omega}_{i,q}\right)^{-1} \cdot \underline{h}\right)^{\leq q} \otimes r\left(\left(\left(\tau^{\omega}_{i,q}\right)^{-1} \cdot \underline{h}\right)_{q+1}\right) \cdot x,$$

where ω *is the IC-sequence of a* q*-simplex,* $\underline{h} \in \mathbf{c}^{q+1}$*, and* $x \in H_p(\mathrm{Hur}^c_{G,(n-q-1)\cdot\underline{\xi}}; A)$*.*

Proof. In combinatorial terms, the differentials $\tilde{\partial}_i$ are given by

$$\left[\left(e, \left(\omega, \sigma L_q\right), \underline{g}\right)\right] \mapsto \left[\left(e, \left(d_i\omega, \sigma\tau^{\omega}_{i,q} L_{q-1}\right), \underline{g}\right)\right],$$

where the $\tau^{\omega}_{q,i}$ are defined as in (4.12). Up to the $\mathrm{Br}_{n\cdot\underline{\xi}}$-action, we may assume that we have $\sigma = 1$, so the map may be written as

$$\left[\left(e, \omega, \underline{g}\right)\right] \mapsto \left[\left(e \cdot \tau^{\omega}_{i,q}, d_i\omega, \left(\tau^{\omega}_{i,q}\right)^{-1} \cdot \underline{g}\right)\right], \tag{5.5}$$

where $[\cdot]$ now denotes L_q- (left) and L_{q-1}-equivalence classes (right), respectively.

Claim: The map (5.5) is L_q-equivariantly homotopic to

$$\left[\left(e, \omega, \underline{g}\right)\right] \mapsto \left[\left(e, d_i\omega, \left(\tau^{\omega}_{i,q}\right)^{-1} \cdot \underline{g}\right)\right]. \tag{5.6}$$

Proof of the claim: Let ι be the identity on $E\mathrm{Br}_{n\cdot\underline{\xi}}$ and τ multiplication in $E\mathrm{Br}_{n\cdot\underline{\xi}}$ by $\tau^{\omega}_{i,q}$. Now, τ descends to a map $B\mathrm{Br}_{n\cdot\underline{\xi}} \to B\mathrm{Br}_{n\cdot\underline{\xi}}$ which is induced by conjugation with $\tau^{\omega}_{i,q}$ in $\mathrm{Br}_{n\cdot\underline{\xi}}$. Since $\tau^{\omega}_{i,q}$ commutes with the elements of L_q by Section 4.2, this conjugation restricts to the identity on L_q. Therefore, both ι and τ descend to self-maps of BL_q homotopic to the identity (note that we may use $E\mathrm{Br}_{n\cdot\underline{\xi}}/L_q$ as a model for BL_q). Hence, τ is L_q-equivariantly freely homotopic to ι. From this fact, the claim follows directly. □

Now, the map

$$\begin{gathered} E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{L_q} \mathbf{c}^n \to E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{L_{q-1}} \mathbf{c}^n \\ \left[\left(e, \underline{g}\right)\right] \mapsto \left[\left(e, \underline{g}\right)\right] \end{gathered}$$

is identified with

$$\mathbf{c}^{q+1} \times \mathrm{Hur}_{G,(n-q-1)\cdot\underline{\xi}} \to \mathbf{c}^{q} \times \mathrm{Hur}_{G,(n-q)\cdot\underline{\xi}},$$

given by left concatenation of a Hurwitz vector with the last ξ-tuple $(\underline{g})_{q+1}$ of $\underline{g} \in \mathbf{c}^n$, where we identify

$$\mathrm{Hur}^c_{G,(n-q-i)\cdot\underline{\xi}} \cong E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{L_{q-i}} \mathbf{c}^{n-q-1}$$

for $i = 0, 1$. In homology, this corresponds to multiplication by $r((\underline{g})_{q+1}) \in R$. Finally, note that $\tau^{\omega}_{i,q}$ only acts on $(\underline{g})^{\leq q+1}$, while L_{q-1} acts on the ξ-tuples $(\underline{g})^{>q+1}$. Combining these facts, taking homology and applying Eilenberg-Zilber to (5.6) yields the desired form of $\overline{\tilde{\partial}_{i*}}$. $\square$

We may now identify the rows of the first page of the spectral sequence from Proposition 5.14 with a class of complexes we can study systematically. In a less general form, the so called $\mathcal{K}$-complexes were introduced in [EVW16, Sect. 4]. Their name stems from their similarity to Koszul complexes.

Definition 5.16. Let M be a graded left R-module. The *$\mathcal{K}$-complex associated to M* is defined as the complex $\mathcal{K}(M)$ with terms

$$\begin{aligned}\mathcal{K}(M)_0 &= M,\\ \mathcal{K}(M)_{q+1} &= A^{l^{\xi}_q} \otimes_A A\langle \mathbf{c}^{q+1}\rangle \otimes_A M(q+1)\end{aligned}$$

for $q = 0, 1, \ldots$, where l^{ξ}_q is given as in Lemma 4.16. The differentials on $\mathcal{K}(M)$ are the linear maps defined by

$$\begin{aligned}\mathrm{d}_{q+1}\colon\ &\mathcal{K}(M)_{q+1} \to \mathcal{K}(M)_q\\ &\omega \otimes \underline{g} \otimes x \mapsto \sum_{i=0}^{q} (-1)^i \left[d_i\omega \otimes \left(\left(\tau^{\omega}_{i,q}\right)^{-1} \cdot \underline{g} \right)^{\leq q} \otimes r\left(\left(\left(\tau^{\omega}_{i,q}\right)^{-1} \cdot \underline{g} \right)_{q+1} \right) \cdot x \right],\end{aligned}$$

where ω is the IC-sequence of a q-simplex, $\underline{g} \in \mathbf{c}^{q+1}$, and $x \in M(q+1)$.

Lemma 5.17. *$\mathcal{K}(M)$ is a complex of graded left R-modules, the grading on $\mathcal{K}(M)_q$ being induced by the grading on $M(q)$. The differential d_q preserves the grading.*

Remark 5.18. For $M = M_p$, the first statement of the lemma is immediate, as we constructed the n-th graded part of the $\mathcal{K}$-complex from the rows of the spectral sequence from Proposition 5.14; cf. also Corollary 5.19.

Proof of Lemma 5.17. The left R-action on $\mathcal{K}(M)_q$ is induced by the left R-module structure on M. Now, the grading on $\mathcal{K}(M)_q$ is induced by the grading on $M(q)$, on which d_q acts by the alternating sum of multiplication with degree one elements. This cancels out with the shifted grading on $\mathcal{K}(M)_{q-1}$.

It remains to check that $\mathcal{K}(M)$ is a complex. Recall that $\mathbf{O}^{[n,\underline{\xi}]}$ is semi-simplicial. Indeed, for $i < j$ and ω the IC-sequence of a q-simplex, we have

$$\begin{aligned}
(d_i d_j \omega, \tau^{\omega}_{j,q} \tau^{d_j\omega}_{i,q-1} L_{q-2}) &= \mathbf{d}_i(d_j\omega, \tau^{\omega}_{j,q} L_{q-1}) \\
&= \mathbf{d}_i \circ \mathbf{d}_j(\omega, L_q) \\
&= \mathbf{d}_{j-1} \circ \mathbf{d}_i(\omega, L_q) \\
&= \mathbf{d}_{j-1}(d_i\omega, \tau^{\omega}_{i,q} L_{q-1}) \\
&= (d_{j-1} d_i \omega, \tau^{\omega}_{i,q} \tau^{d_i\omega}_{j-1,q-1} L_{q-2}).
\end{aligned}$$

Thus, we obtain

$$d_i d_j \omega = d_{j-1} d_i \omega \tag{5.7}$$

as well as $\tau^{\omega}_{j,q} \tau^{d_j\omega}_{i,q-1} L_{q-2} = \tau^{\omega}_{i,q} \tau^{d_i\omega}_{j-1,q-1} L_{q-2}$. In particular, since L_{q-2} acts trivially on the first $(q-1)\xi$ entries of tuples $\underline{g} \in \mathbf{c}^{q+1}$, we have

$$\left(\left(\tau^{\omega}_{j,q} \tau^{d_j\omega}_{i,q-1}\right)^{-1} \cdot \underline{g}\right)^{\leq q-1} = \left(\left(\tau^{\omega}_{i,q} \tau^{d_i\omega}_{j-1,q-1}\right)^{-1} \cdot \underline{g}\right)^{\leq q-1}. \tag{5.8}$$

as well as

$$\begin{aligned}
&r\left(\left(\left(\tau^{\omega}_{j,q} \tau^{d_j\omega}_{i,q-1}\right)^{-1} \cdot \underline{g}\right)_q\right) \cdot r\left(\left(\left(\tau^{\omega}_{j,q} \tau^{d_j\omega}_{i,q-1}\right)^{-1} \cdot \underline{g}\right)_{q+1}\right) \\
&\quad = r\left(\left(\left(\tau^{\omega}_{i,q} \tau^{d_i\omega}_{j-1,q-1}\right)^{-1} \cdot \underline{g}\right)_q\right) \cdot r\left(\left(\left(\tau^{\omega}_{i,q} \tau^{d_i\omega}_{j-1,q-1}\right)^{-1} \cdot \underline{g}\right)_{q+1}\right).
\end{aligned} \tag{5.9}$$

Here, we introduced the inversion for convenience in the next steps. Furthermore, the relation

$$\left(\left(\tau^{\omega}_{i,q}\right)^{-1} \cdot \underline{g}\right)_{q+1} = \left(\left(\tau^{\omega}_{i,q} \tau^{d_i\omega}_{j-1,q-1}\right)^{-1} \cdot \underline{g}\right)_{q+1} \tag{5.10}$$

is valid, as $\tau^{d_i\omega}_{j-1,q-1}$ acts nontrivially only on the first $q\xi$ entries of a tuple $\underline{g}$.

Indeed, we may now finish the proof by showing that $\mathrm{d}_q \circ \mathrm{d}_{q+1} = 0$ holds. Let $\omega \otimes \underline{g} \otimes x$ be an elementary tensor in $\mathcal{K}(M)_{q+1}$. Now, we see

$$(\mathrm{d}_q \circ \mathrm{d}_{q+1})(\omega \otimes \underline{g} \otimes x)$$

$$
\begin{aligned}
&= \mathrm{d}_q\left(\sum_{i=0}^{q}(-1)^i\left[d_i\omega\otimes\left(\left(\tau_{i,q}^{\omega}\right)^{-1}\cdot\underline{g}\right)^{\leq q}\otimes r\left(\left(\left(\tau_{i,q}^{\omega}\right)^{-1}\cdot\underline{g}\right)_{q+1}\right)\cdot x\right]\right)\\
&\overset{(5.10)}{=} \sum_{i=0}^{q}(-1)^i\left[\sum_{j=0}^{q-1}(-1)^j\left[d_jd_i\omega\otimes\left(\left(\tau_{i,q}^{\omega}\tau_{j,q-1}^{d_i\omega}\right)^{-1}\cdot\underline{g}\right)^{\leq q-1}\right.\right.\\
&\qquad\left.\left.\otimes\, r\left(\left(\left(\tau_{i,q}^{\omega}\tau_{j,q-1}^{d_i\omega}\right)^{-1}\cdot\underline{g}\right)_{q}\right)\cdot r\left(\left(\left(\tau_{i,q}^{\omega}\tau_{j,q-1}^{d_i\omega}\right)^{-1}\cdot\underline{g}\right)_{q+1}\right)\cdot x\right]\right]\\
&= \sum_{0\leq j<i\leq q}(-1)^{i+j}\left[d_jd_i\omega\otimes\left(\left(\tau_{i,q}^{\omega}\tau_{j,q-1}^{d_i\omega}\right)^{-1}\cdot\underline{g}\right)^{\leq q-1}\right.\\
&\qquad\left.\otimes\, r\left(\left(\left(\tau_{i,q}^{\omega}\tau_{j,q-1}^{d_i\omega}\right)^{-1}\cdot\underline{g}\right)_{q}\right)\cdot r\left(\left(\left(\tau_{i,q}^{\omega}\tau_{j,q-1}^{d_i\omega}\right)^{-1}\cdot\underline{g}\right)_{q+1}\right)\cdot x\right]\\
&\quad+\sum_{0\leq i<j\leq q}(-1)^{i+j-1}\left[d_{j-1}d_i\omega\otimes\left(\left(\tau_{i,q}^{\omega}\tau_{j-1,q-1}^{d_i\omega}\right)^{-1}\cdot\underline{g}\right)^{\leq q-1}\right.\\
&\qquad\left.\otimes\, r\left(\left(\left(\tau_{i,q}^{\omega}\tau_{j-1,q-1}^{d_i\omega}\right)^{-1}\cdot\underline{g}\right)_{q}\right)\cdot r\left(\left(\left(\tau_{i,q}^{\omega}\tau_{j-1,q-1}^{d_i\omega}\right)^{-1}\cdot\underline{g}\right)_{q+1}\right)\cdot x\right]\\
&\underset{(5.8),\,(5.9)}{\overset{(5.7)}{=}} \sum_{0\leq j<i\leq q}(-1)^{i+j}\left[d_jd_i\omega\otimes\left(\left(\tau_{i,q}^{\omega}\tau_{j,q-1}^{d_i\omega}\right)^{-1}\cdot\underline{g}\right)^{\leq q-1}\right.\\
&\qquad\left.\otimes\, r\left(\left(\left(\tau_{i,q}^{\omega}\tau_{j,q-1}^{d_i\omega}\right)^{-1}\cdot\underline{g}\right)_{q}\right)\cdot r\left(\left(\left(\tau_{i,q}^{\omega}\tau_{j,q-1}^{d_i\omega}\right)^{-1}\cdot\underline{g}\right)_{q+1}\right)\cdot x\right]\\
&\quad+\sum_{0\leq i<j\leq q}(-1)^{i+j-1}\left[d_id_j\omega\otimes\left(\left(\tau_{j,q}^{\omega}\tau_{i,q-1}^{d_j\omega}\right)^{-1}\cdot\underline{g}\right)^{\leq q-1}\right.\\
&\qquad\left.\otimes\, r\left(\left(\left(\tau_{j,q}^{\omega}\tau_{i,q-1}^{d_j\omega}\right)^{-1}\cdot\underline{g}\right)_{q}\right)\cdot r\left(\left(\left(\tau_{j,q}^{\omega}\tau_{i,q-1}^{d_j\omega}\right)^{-1}\cdot\underline{g}\right)_{q+1}\right)\cdot x\right]=0,
\end{aligned}
$$

which proves the lemma. □

Corollary 5.19. *For the R-module M_p from Notation 3.18, there is a homological spectral sequence $E^1_{qp} = [\mathcal{K}(M_p)_{q+1}]_n$ with differentials given by the differentials on $\mathcal{K}(M_p)$ which converges to $H_{p+q}(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}};A)$ for $p+q\leq\lfloor\frac{n}{2}\rfloor-2$.*

Proof. By construction, we have

$$
[\mathcal{K}(M_p)_q]_n = A^{l^{\xi}_{q+1}}\otimes_A A\langle\mathbf{c}^{q+1}\rangle\otimes_A H_p(\mathrm{Hur}^c_{(n-q-1)\cdot\underline{\xi}};A)=\bar{E}^1_{qp}
$$

for $q<n$, and $[\mathcal{K}(M_p)_{q+1}]_n = 0$ otherwise. The statement follows directly from Proposition 5.14, Lemma 5.15, and the construction of the differential on $\mathcal{K}(M)$. □

We may now take a step towards the homology of $\mathcal{K}$-complexes, where $M=M_0=R$.

In this case, multiplication in R gives $\mathcal{K}(R)_q$ (and hence also the homology modules of $\mathcal{K}(R)$) the structure of a two-sided graded R-module. A simpler version of the next lemma is proved in [EVW16, Lemma 4.11].

Lemma 5.20. *For all $q \geq 0$, $H_q(\mathcal{K}(R))$ is killed by the right action of $R_{>0}$.*

Proof. For simplicity of notation, in this proof we work with P-sequences instead of IC-sequences, cf. Section 4.2 (and in particular Remark 4.24).

Recall that for a Hurwitz vector $\underline{g} \in \mathbf{c}^{q+1}$, we write $\partial\underline{g}$ for its boundary, which is invariant under the $\mathrm{Br}_{(q+1)\cdot\underline{\xi}}$-action. By Remark 3.15, R is generated as an A-module by orbits $[\underline{s}] \in \mathbf{c}^n \,/\, \mathrm{Br}_{n\cdot\underline{\xi}}$, for $n \geq 0$. Let $h \in \mathbf{c}$, such that the elements of the form $r(h)$ generate $R_{>0}$ as an A-algebra. We define a map $S_h\colon\, \mathcal{K}(R)_{q+1} \to \mathcal{K}(R)_{q+2}$ by linear extension of

$$S_h(\tilde{\omega} \otimes \underline{g} \otimes [\underline{s}]) = (\xi + \tilde{\omega}) \otimes \left(h^{\left(\partial\underline{g}\partial[\underline{s}]\right)^{-1}}, \underline{g}\right) \otimes [\underline{s}],$$

where $\tilde{\omega}$ is the P-sequence of a q-simplex, and $\underline{g}, h, \underline{s}$ as above. Here, $(\xi + \tilde{\omega})$ denotes the P-sequence of a $(q+1)$-simplex obtained by increasing every entry of $\tilde{\omega}$ by ξ and then attaching $(1, \ldots, \xi)$ from the left. In particular, the equations

$$\begin{aligned} d_0(\xi + \tilde{\omega}) &= \tilde{\omega} \\ d_i(\xi + \tilde{\omega}) &= (\xi + d_{i-1}\tilde{\omega}) \end{aligned} \tag{5.11}$$

hold for $i = 1, \ldots, q$. Furthermore, since the first ξ positions of the P-sequence $\xi + \tilde{\omega}$ are given by $1, \ldots, \xi$, the equations

$$\begin{aligned} \left(\tau_{0,q+1}^{(\xi+\tilde{\omega})}\right)^{-1} \cdot \left(h, \underline{g}\right) &= \left(\underline{g}, h^{\partial\underline{g}}\right) \\ \left(\tau_{i+1,q+1}^{(\xi+\tilde{\omega})}\right)^{-1} \cdot \left(h, \underline{g}\right) &= \left(h, \left(\tau_{i,q}^{\tilde{\omega}}\right)^{-1} \cdot \underline{g}\right) \end{aligned} \tag{5.12}$$

are true for any $h \in \mathbf{c}$ and $i = 0, \ldots, q$, cf. the definition of $\tau_{i,q}^{\tilde{\omega}}$ in (4.12).

We claim that S_h is a chain homotopy from right multiplication with $r(h)$ to the zero map. Indeed, we have

$$\begin{aligned} &(\mathrm{d}_{q+1}\, S_h + S_h\, \mathrm{d}_q)\left(\tilde{\omega} \otimes \underline{g} \otimes [\underline{s}]\right) \\ &\quad= \mathrm{d}_{q+1}\left((\xi + \tilde{\omega}) \otimes \left(h^{\left(\partial\underline{g}\partial\underline{s}\right)^{-1}}, \underline{g}\right) \otimes [\underline{s}]\right) \\ &\qquad + S_h\left(\sum_{i=0}^{q}(-1)^i\left[d_i\tilde{\omega} \otimes \left(\left(\tau_{i,q}^{\tilde{\omega}}\right)^{-1} \cdot \underline{g}\right)^{\leq q} \otimes \left[\left(\left(\left(\tau_{i,q}^{\tilde{\omega}}\right)^{-1} \cdot \underline{g}\right)_{q+1}, \underline{s}\right)\right]\right]\right) \end{aligned}$$

$$
\begin{aligned}
&\overset{(5.11)}{\underset{(5.12)}{=}} \tilde{\omega} \otimes \underline{g} \otimes \left(h^{(\partial \underline{s})^{-1}}, [\underline{s}]\right) \\
&\quad + \sum_{i=0}^{q} (-1)^{i+1} \Bigg[(\xi + d_i \tilde{\omega}) \\
&\qquad \otimes \left(h^{\left(\partial \underline{g} \partial \underline{s}\right)^{-1}}, \left(\left(\tau_{i,q}^{\tilde{\omega}}\right)^{-1} \cdot \underline{g} \right)^{\leq q} \right) \otimes \left[\left(\left(\left(\tau_{i,q}^{\tilde{\omega}}\right)^{-1} \cdot \underline{g} \right)_{q+1}, \underline{s} \right) \right] \Bigg] \\
&\quad + \sum_{i=0}^{q} (-1)^{i} \Bigg[(\xi + d_i \tilde{\omega}) \\
&\qquad \otimes \left(h^{\left(\partial\left(\left(\tau_{i,q}^{\tilde{\omega}}\right)^{-1} \cdot \underline{g}\right) \partial \underline{s}\right)^{-1}}, \left(\left(\tau_{i,q}^{\tilde{\omega}}\right)^{-1} \cdot \underline{g} \right)^{\leq q} \right) \otimes \left[\left(\left(\left(\tau_{i,q}^{\tilde{\omega}}\right)^{-1} \cdot \underline{g} \right)_{q+1}, \underline{s} \right) \right] \Bigg] \\
&= \tilde{\omega} \otimes \underline{g} \otimes [\underline{s} \cdot r(h)],
\end{aligned}
$$

as $\left(h^{(\partial \underline{s})^{-1}}, \underline{s}\right)$ is equivalent to $(\underline{s}, h)$ under the $\mathrm{Br}_{(n+1)\cdot\underline{\xi}}$-action. Hence, S_h is the desired chain homotopy. □

5.4. Modules over stabilized rings

This section is mainly a detailed review of [EVW16, Sect. 4] with some adjustments to our setup. We work with graded modules over graded rings satisfying a specific stability condition which will eventually form the essential criterion for homological stabilization. These definitions generalize the modules M_p over the ring R.

More specifically, notable changes to the lines of argument from [EVW16, Sect. 4] include the proof of Lemma 5.24, the more general statement of Lemma 5.25, the less general statement of Lemma 5.30, and the improved stable ranges in Proposition 5.34. Most proofs are worked out in slightly more detail than in the original article.

Definition 5.21. Let $R = \bigoplus_{i \in \mathbb{N}_0} R_i$ be a graded ring, $A = R/R_{>0} \cong R_0$ the ring of degree zero elements, and $U \in R$ a central homogeneous element of positive degree. The ring R is called *A-stabilized by U* if the three following conditions are satisfied:

(i) Both kernel and cokernel of the multiplication $R \xrightarrow{U\cdot} R$ have finite degree as graded R-modules; in other words, $D_R(U) = \max\{\deg(R/UR), \deg(R[U])\}$ is finite.

(ii) A is a commutative ring.

(iii) R is generated in degree one (i.e., by R_1) as an algebra over A.

We call U the *stabilizing element* for R.

Remark 5.22. Note that due to the centrality of U, it does not matter whether multiplication with U is taken from the left or from the right.

The definition of stabilized rings is inspired by properties of the ring of connected components of Hurwitz spaces. We have already seen above that these rings satisfy condition (iii). Furthermore, we fixed A to be commutative.

Hence, the main feature of stabilized rings turns out to be condition (i). It implies that multiplication by U is an isomorphism of abelian groups $R_N \cong R_{N+\deg U}$ for N large enough. We show in the rest of this chapter that if the ring of connected components is A-stabilized by U, $\deg U$-periodic homological stability (with A-coefficients) for Hurwitz spaces follows automatically, which is the claim of Theorem 5.7.

For the rest of Section 5.4, let R be A-stabilized by U. We are concerned with homological properties of R-modules.

5.4.1. Homology of graded modules

From the multiplication in R, the ring $A = R/R_{>0}$ inherits the structure of an R-bimodule.

Notation 5.23. For a graded left R-module M, we write $H_i(M)$ for the graded left R-module $\mathrm{Tor}_i^R(A, M)$, so $H_0(M) = M/R_{>0}M$; accordingly, $H_i(N) = \mathrm{Tor}_i^R(N, A)$ for a graded right R-module N.

Furthermore, we write $\bar{R} = R/UR$.

Lemma 5.24. *For M a graded left R-module and N a graded right R-module, we have* $\deg(N \otimes_R M) \leq \min\{\deg(N) + \deg(H_0(M)), \deg(H_0(N)) + \deg(M)\}$.

Proof. We prove the first inequality, the second one is handled analogously. If either $\deg(N)$ or $\deg(H_0(M))$ is infinite, the claim is clearly true, and we assume similar finiteness conditions in the following without further mention. We may thus assume that both degrees are finite. As the tensor product of modules is generated as an abelian group by pure tensors, it suffices to prove the inequality for tensors of the form $n \otimes m$, where $n \in N, m \in M$.

As an R-module, M is generated in degree at most $\deg(H_0(M)) = \deg(M/R_{>0}M)$. For an element $m \in M$, we write $m = \sum_{\alpha\in\mathcal{I}} \lambda_\alpha m_\alpha$, where $\mathcal{I}$ is some indexing set, $\{m_\alpha\}_{\alpha\in\mathcal{I}} \subset H_0(M)$ generates M as a left R-module, and $\lambda_\alpha \in R$ for all $\alpha \in \mathcal{I}$.

Then, for $n \in N$ and m as above, we obtain

$$\begin{aligned}
\deg(n \otimes m) &= \deg\left(n \otimes \sum_{\alpha \in \mathcal{I}} \lambda_\alpha m_\alpha\right) \\
&= \deg\left(\sum_{\alpha \in \mathcal{I}} (\lambda_\alpha n \otimes m_\alpha)\right) \\
&\leq \max_{\alpha \in \mathcal{I}} \{\deg(\lambda_\alpha n \otimes m_\alpha)\} \\
&\leq \deg(N) + \deg(H_0(M)),
\end{aligned}$$

as claimed. $\square$

Lemma 5.25. *Let $\bar{M}$ be a graded left $\bar{R} = R/UR$-module and $\bar{N}$ a graded right $\bar{R}$-module. Then,* $\deg(\mathrm{Tor}_i^{\bar{R}}(\bar{N}, \bar{M})) \leq i \deg(\bar{R}) + \deg(\bar{N}) + \deg(\bar{M})$.

Proof. We construct a projective resolution

$$\ldots \to P_2 \to P_1 \to P_0 \to \bar{N} \to 0$$

of $\bar{N}$ by free graded right $\bar{R}$-modules: Choose P_i as the free right $\bar{R}$-module on a set of generators for $\ker(P_{i-1} \to P_{i-2})$ as a right $\bar{R}$-module, where $P_{-1} = \bar{N}$ and $P_{-2} = 0$. By this construction, P_i is generated as an $\bar{R}$-module in degree at most $i \deg(\bar{R}) + \deg(\bar{N})$, so $\deg(H_0(P_i)) = \deg(P_i / R_{>0} P_i) \leq i \deg(\bar{R}) + \deg(\bar{N})$.

Now, Lemma 5.24 gives

$$\begin{aligned}
\deg(\mathrm{Tor}_i^{\bar{R}}(\bar{N}, \bar{M})) &\leq \deg(P_i \otimes_{\bar{R}} \bar{M}) \\
&\leq \deg(H_0(P_i)) + \deg(\bar{M}) \\
&\leq i \deg(\bar{R}) + \deg(\bar{N}) + \deg(\bar{M}),
\end{aligned}$$

which is what we wanted to show. $\square$

Notation 5.26. Let M be a graded left R-module. We define:

$$\begin{aligned}
D_M(U) &= \max\{\deg(M[U]), \deg(M/UM)\} \\
\delta_M(U) &= \max\{\deg(\mathrm{Tor}_0^R(\bar{R}, M)), \deg(\mathrm{Tor}_1^R(\bar{R}, M))\}
\end{aligned}$$

These constants may be infinite except for $D_R(U)$, which is always finite because R is A-stabilized by U.

Though $D_M(U)$ and $\delta_M(U)$ depend heavily on the stabilizing element $U \in R$, we usually use the symbols D_M and δ_M if U is fixed and/or clear from the context.

Lemma 5.27. *For any graded left R-module M, $D_M \leq \delta_M + D_R$.*

Proof. By definition, $\mathrm{Tor}_0^R(\bar{R}, M) = \bar{R} \otimes_R M \cong M/UM$, so the assertion is clear for $\deg(M/UM)$. We now have to bound the degree of $M[U]$.

In the following, we show that there is a sequence

$$R[U] \otimes_R M \to M[U] \to \mathrm{Tor}_1^R(\bar{R}, M) \tag{5.13}$$

which is exact in the middle, where the first map is of degree zero and the second one of degree $\deg(U)$. The existence of this sequence implies the statement of the lemma: By exactness in the middle, we obtain

$$\deg(M[U]) \leq \max\{\deg(R[U] \otimes_R M), \deg(\mathrm{Tor}_1^R(\bar{R}, M)) - \deg U\}.$$

Now $\deg(\mathrm{Tor}_1^R(\bar{R}, M)) - \deg U \leq \delta_M + D_R$ is true by definition, while Lemma 5.24 yields

$$\begin{aligned} \deg(R[U] \otimes_R M) &\leq \deg(H_0(M)) + \deg(R[U]) \\ &\leq \deg(M/UM) + \deg(R[U]) \\ &\leq \delta_M + D_R. \end{aligned}$$

That is, we have to prove the existence of (5.13). We write UR for the two-sided ideal in R generated by U and consider the graded left R-module $N = (UR) \otimes_R M$. Multiplication by U induces a map $M \xrightarrow{U\cdot} M$ of degree $\deg U$ which factors as

$$M \xrightarrow{\alpha} N \xrightarrow{\beta} M,$$

where $\alpha(m) = U \otimes m$, and $\beta(r \otimes m) = r \cdot m$, so α is of degree $\deg U$ and β of degree zero. This factorization gives an exact sequence

$$0 \to \ker(\alpha) \to M[U] \xrightarrow{\alpha} \ker(\beta)$$

since $M[U] = \ker(\beta \circ \alpha)$.

The exact sequence

$$0 \to UR \to R \to \bar{R} \to 0$$

is a projective resolution of $\bar{R}$ as an R-module; tensoring with M, we obtain

$$N \xrightarrow{\beta} M \to M/UM \to 0,$$

so $\mathrm{Tor}_1^R(\bar{R}, M) \cong \ker(\beta)$.

Similarly, from the exact sequence

$$0 \to R[U] \to R \xrightarrow{U\cdot} UR \to 0,$$

we gain

$$R[U] \otimes_R M \to M \xrightarrow{\alpha} N \to 0$$

from tensoring with M, so there is a surjection $R[U] \otimes_R M \twoheadrightarrow \ker(\alpha)$.

Combining the two last facts, we have a sequence

$$R[U] \otimes_R M \twoheadrightarrow \ker(\alpha) \hookrightarrow M[U] \xrightarrow{\alpha} \ker(\beta) \cong \mathrm{Tor}_1^R(\bar{R}, M), \tag{5.14}$$

which is exact at $M[U]$. Omitting $\ker(\alpha)$ yields the desired sequence (5.13), which is still exact at $M[U]$ due to the surjectivity of the first map in (5.14). □

Lemma 5.28. *For a graded left R-module M and $i \geq 0$,* $\deg(\mathrm{Tor}_i^R(\bar{R}, M)) \leq \delta_M + D_R i$.

Proof. From the definition of δ_M, the cases $i = 0$ and $i = 1$ are immediate. We proceed by induction on i.

The subring $R[U]$ is a right $\bar{R}$-module. We construct a resolution of $R[U]$ by free right $\bar{R}$-modules

$$\ldots \to P_4 \to P_3 \to P_2 \to R[U] \to 0$$

analogous to the resolution in the proof of Lemma 5.25. Hence, P_i is generated in degrees not greater than $\deg(R[U]) + (i-2)\deg(\bar{R}) \leq (i-1)D_R$. We combine the resolution with

$$R[U] \hookrightarrow R \xrightarrow{U\cdot} R \twoheadrightarrow \bar{R} \to 0$$

and obtain a resolution of $\bar{R}$ by right R-modules (as any $\bar{R}$-module is also an R-module)

$$\ldots \to P_4 \to P_3 \to P_2 \to R \to R \to \bar{R} \to 0,$$

so $P_1 = P_0 = R$. Now, there is a hyperhomology spectral sequence (cf. (B.4))

$$E^1_{ij} = \mathrm{Tor}_i^R(P_j, M) \Longrightarrow \mathrm{Tor}_{i+j}^R(\bar{R}, M).$$

Therefore, for $k \geq 2$, $\mathrm{Tor}_k^R(\bar{R}, M)$ is filtered by subquotients of

$$\bigoplus_{j=0}^{k} \mathrm{Tor}_j^R(P_{k-j}, M) = \bigoplus_{j=0}^{k-2} \mathrm{Tor}_j^R(P_{k-j}, M),$$

where the last equation is due to the fact that $\mathrm{Tor}_i^R(R, M) = 0$ for $i > 0$. For $i \geq 2$, we chose P_i as a free $\bar{R}$-module generated in degree at most $(i-1)D_R$. Thus, for the degrees of the summands, we have

$$\begin{aligned}
\deg(\mathrm{Tor}_j^R(P_{k-j}, M)) &\leq \deg(\mathrm{Tor}_j^R(\bar{R}((k-j-1)D_R), M)) \\
&\leq (k-j-1)D_R + \deg(\mathrm{Tor}_j^R(\bar{R}, M)) \\
&\overset{(*)}{\leq} (k-j-1)D_R + \delta_M + jD_R \\
&\leq \delta(M) + kD_R
\end{aligned}$$

for $j = 0, \ldots, k-2$. Here, step $(*)$ follows from the induction hypothesis, as $j < k$. This finally implies

$$\begin{aligned}
\deg(\mathrm{Tor}_k^R(\bar{R}, M)) &\leq \deg\left(\bigoplus_{j=0}^{k-2} \mathrm{Tor}_j^R(P_{k-j}, M)\right) \\
&= \max_{0 \leq j \leq k-2}\{\deg(\mathrm{Tor}_j^R(P_{k-j}, M))\} \\
&\leq \delta_M + kD_R,
\end{aligned}$$

which is what we wanted to show. $\square$

Lemma 5.29. *For a left $\bar{R}$-module $\bar{M}$, we have* $\deg(\bar{M}) \leq \deg(A \otimes_{\bar{R}} \bar{M}) + \deg(\bar{R})$.

Proof. Choose a set of homogeneous elements $\{x_i \mid i \in \mathcal{I}\}$ projecting to a generating set for $A \otimes_{\bar{R}} \bar{M}$ as an A-module. Next, we consider the quotient $Q = \bar{M}/(\sum_i \bar{R}x_i)$ which is a graded $\bar{R}$-module by homogeneity of the x_i. In particular, we have $A \otimes_{\bar{R}} Q = 0$. Indeed, the short exact sequence of $\bar{R}$-modules

$$0 \to \sum_{i \in \mathcal{I}} \bar{R}x_i \to \bar{M} \to Q \to 0$$

gives an exact sequence of A-modules

$$A \otimes_{\bar{R}} \left(\sum_{i \in \mathcal{I}} \bar{R}x_i\right) \to A \otimes_{\bar{R}} \bar{M} \to A \otimes_{\bar{R}} Q \to 0.$$

The first map is surjective since the x_i generate $A \otimes_{\bar{R}} \bar{M}$ as an A-module; thence, $A \otimes_{\bar{R}} Q = 0$.

This last statement can be reformulated as $Q = R_{>0}Q$, since we have $A = R/R_{>0}$. Now, this holds if and only if $Q = 0$: The degrees of the graded parts of the module Q are bounded from below, and by the action of $R_{>0}$, the degree of the component of lowest degree is shifted upwards (this reasoning is also known as the Graded Nakayama Lemma).

Hence, $\bar{M}$ is generated as an $\bar{R}$-module by the x_i. As $\deg(x_i) \leq \deg(A \otimes_{\bar{R}} \bar{M})$, this yields the assertion. □

Lemma 5.30. *If M is a graded left R-module,*

$$\delta_M \leq \max\{\deg(H_0(M)), \deg(H_1(M))\} + 4D_R.$$

Proof. By Lemma 5.24,

$$\begin{aligned}\deg(\mathrm{Tor}_0^R(\bar{R}, M)) &= \deg(\bar{R} \otimes_R M) \\ &\leq \deg(H_0(M)) + \deg(\bar{R}) \\ &\leq \deg(H_0(M)) + D_R.\end{aligned} \tag{5.15}$$

We have yet to bound the degree of $\mathrm{Tor}_1^R(\bar{R}, M)$. The base change spectral sequence for Tor (cf. (B.5)),

$$E_{ij}^2 = \mathrm{Tor}_i^{\bar{R}}(A, \mathrm{Tor}_j^R(\bar{R}, M)) \Longrightarrow \mathrm{Tor}_{i+j}^R(A, M),$$

yields the existence of a sequence

$$\mathrm{Tor}_2^{\bar{R}}(A, \bar{R} \otimes_R M) \to A \otimes_{\bar{R}} \mathrm{Tor}_1^R(\bar{R}, M) \to H_1(M) \tag{5.16}$$

which is exact in the middle: It is given by

$$E_{20}^2 \xrightarrow{d_2} E_{01}^2 \to \mathrm{Tor}_1^R(A, M),$$

where the second map is an edge map in the spectral sequence. Now, we have

$$E_{01}^2 / d_2(E_{20}^2) = E_{01}^3 = E_{01}^\infty \hookrightarrow \mathrm{Tor}_1^R(A, M),$$

which implies the desired exactness.

Furthermore, by applying Lemma 5.29 and the exactness of (5.16), we obtain

$$\begin{aligned}\deg(\operatorname{Tor}_1^R(\bar{R}, M)) &\leq \deg(A \otimes_{\bar{R}} \operatorname{Tor}_1^R(\bar{R}, M)) + \deg(\bar{R}) \\ &\leq \max\{\deg(\operatorname{Tor}_2^{\bar{R}}(A, \bar{R} \otimes_R M)), \deg(H_1(M))\} + \deg(\bar{R}),\end{aligned} \tag{5.17}$$

while Lemma 5.25 and the inequality (5.15) give

$$\begin{aligned}\deg(\operatorname{Tor}_2^{\bar{R}}(A, \bar{R} \otimes_R M)) &\leq 2\deg(\bar{R}) + \deg(\bar{R} \otimes_R M) \\ &\leq 3\deg(\bar{R}) + \deg(H_0(M)).\end{aligned}$$

In combination with (5.17) and $\deg(\bar{R}) \leq D_R$, this implies the lemma. ☐

By [EVW16, Lemma 4.10], Lemma 5.31 is also true for a right R-module $\bar{N}$. We chose to leave it at the case proved below which suffices for the application in the proof of Lemma 5.33.

Lemma 5.31. *For a graded left R-module M and a graded right $\bar{R}$-module $\bar{N}$, the following inequality is valid:*

$$\deg(\operatorname{Tor}_i^R(\bar{N}, M)) \leq \deg(\bar{N}) + \max\{\deg(H_0(M)), \deg(H_1(M))\} + (4 + i) \cdot D_R.$$

Proof. We make successive use of the previous lemmas' statements, as well as the base change spectral sequence for Tor in the first step:

$$\begin{aligned}\deg(\operatorname{Tor}_i^R(\bar{N}, M)) \ &\leq \ \max_{a+b=i}\{\deg(\operatorname{Tor}_a^{\bar{R}}(\bar{N}, \operatorname{Tor}_b^R(\bar{R}, M)))\} \\ &\overset{5.25}{\leq} \max_{a+b=i}\{\deg(\bar{R}) \cdot a + \deg(\bar{N}) + \deg(\operatorname{Tor}_b^R(\bar{R}, \bar{N}))\} \\ &\overset{5.28}{\leq} D_R i + \deg(\bar{N}) + \delta_M \\ &\overset{5.30}{\leq} \deg(\bar{N}) + \max\{\deg(H_0(M)), \deg(H_1(M))\} + (4 + i)D_R,\end{aligned}$$

where in the third step we also used the fact $\deg(\bar{R}) \leq D_R$. ☐

5.4.2. Homology of $\mathcal{K}$-complexes

The theory of $\mathcal{K}$-complexes introduced in Section 5.3 may now be merged with this section's homological results about A-stabilized rings R. From now on, we assume that D_R is positive (cf. Remark 5.6).

Lemma 5.32. $\deg(H_q(\mathcal{K}(R))) \leq D_R + \deg U + q$.

Proof. By definition, $\mathcal{K}(R)_q$ is a direct sum of copies of the shift $R[q]$. As R is A-stabilized by U, the endomorphism of $\mathcal{K}(R)$ induced by multiplication with U is an isomorphism in source degree $n > D_R + q$. Since U is central, this map restricts to a map $U\colon \ker \mathrm{d}_q \to \ker \mathrm{d}_q$ of kernels which is an isomorphism in the same range. Therefore, $\ker \mathrm{d}_q$ is generated as a right R-module in degree at most $D_R+q+\deg U$, and so the same holds for its quotient $H_q(\mathcal{K}(R))$. Now, Lemma 5.20 states that $H_q(\mathcal{K}(R))$ is killed by the right action of $R_{>0}$, thus $\deg(H_q(\mathcal{K}(R))) \leq D_R + q + \deg U$. □

Lemma 5.33. *For a graded left R-module M,*

$$\deg(H_q(\mathcal{K}(M))) \leq \max\{\deg(H_0(M)), \deg(H_1(M))\} + (q+5)\cdot D_R + \deg U$$

holds for all $q \geq 0$.

Proof. We have $\mathcal{K}(M) = \mathcal{K}(R) \otimes_R M$; furthermore, $\mathcal{K}(R)_q$ is a free R-module for all $q \geq 0$. We may therefore apply the universal coefficient spectral sequence (cf. (B.3)) in the following form:

$$E^2_{i,q-i} = \mathrm{Tor}_i^R\left(H_{q-i}(\mathcal{K}(R)), M\right) \Longrightarrow H_q(\mathcal{K}(M)).$$

Thus, $H_q(\mathcal{K}(M))$ admits a filtration by subquotients of $\bigoplus_{i=0}^q \mathrm{Tor}_i^R\left(H_{q-i}(\mathcal{K}(R)), M\right)$, so the preceeding lemmas imply

$$\begin{aligned}
&\deg(H_q(\mathcal{K}(M)))\\
&\leq \max_{i=0,\dots,q}\{\deg(\mathrm{Tor}_i^R(H_{q-i}(\mathcal{K}(R)), M))\}\\
&\overset{5.31}{\leq} \max_{i=0,\dots,q}\{\deg(H_{q-i}(\mathcal{K}(R))) + \max\{\deg(H_0(M)), \deg(H_1(M))\} + (4+i)\cdot D_R\}\\
&\overset{5.32}{\leq} \max\{\deg(H_0(M)), \deg(H_1(M))\} + (q+5)\cdot D_R + \deg U,
\end{aligned}$$

which shows the lemma: The fact from Lemma 5.20 that $H_{q-i}(\mathcal{K}(R))$ is killed by the right $R_{>0}$-action gives $H_{q-i}(\mathcal{K}(R))$ the structure of a right $\bar{R}$-module, which enables us to apply Lemma 5.31. In addition, the assumption $D_R \geq 1$ was used in the last estimate. □

The upcoming proposition (cf. also [EVW16, Thm. 4.2]) combines the results from the preceeding lemmas and provides an important ingredient for the proof of Theorem 5.7.

Proposition 5.34. *Let M be a graded left R-module and $h_i = \deg(H_i(\mathcal{K}(M)))$. Then, we have $h_q \leq \max\{h_0, h_1\} + D_R q + (5D_R + \deg U)$, and multiplication by U, $M \xrightarrow{U\cdot} M$, is an isomorphism in source degree greater than or equal to $\max\{h_0, h_1\} + 5D_R + 1$.*

Proof. We show that for $i = 0, 1$,

$$\deg(H_i(M)) \leq h_i. \tag{5.18}$$

Using this result, we obtain

$$\begin{aligned} h_q &\overset{5.33}{\leq} \max\{\deg(H_0(M)), \deg(H_1(M))\} + (q+5) \cdot D_R + \deg U \\ &\overset{(5.18)}{\leq} \max\{h_0, h_1\} + D_R q + (5D_R + \deg U), \end{aligned}$$

which is the first part of the statement.

Furthermore, by Lemma 5.27, multiplication by U is an isomorphism in source degree greater or equal $\delta_M + D_R + 1$. This expression is by Lemma 5.30 and (5.18) bounded from above by $\max\{h_0, h_1\} + 5D_R + 1$, which gives the second claim of the proposition.

It remains to show (5.18). For $i = 0$, we have $H_0(M) = A \otimes_R M = M/R_{>0}M$ and $H_0(\mathcal{K}(M)) = M/\operatorname{im} \mathrm{d}_1$. Now, $\mathrm{d}_1(\omega, g, x) = r(g) \cdot x$ for all elementary tensors in $\mathcal{K}(M)_1$, so $\operatorname{im} \mathrm{d}_1 = R_{>0}M$ and the claim is vacuously true.

For $i = 1$, we factor the map $\mathrm{d}_1 \colon \mathcal{K}(M)_1 \to M$ as $\mathrm{d}_1 = \beta \circ \alpha$,

$$\mathcal{K}(M)_1 = A^{l_0^\xi} \otimes_A A\langle \mathbf{c} \rangle \otimes_A M(1) \xrightarrow{\alpha} R_{>0} \otimes_R M \xrightarrow{\beta} M,$$

with $\alpha(\omega \otimes g \otimes x) = r(g) \otimes x$ and $\beta(r \otimes x) = r \cdot x$. As $R_{>0}$ is generated by elements of the form $r(g)$, we can factor any $r \in R_{>0}$ as $r = r(g) \cdot r'$ for some $r' \in R$; therefore, α is surjective. It is also degree-preserving – note that $R_{>0} \otimes_R M$ is graded via $\deg(r \otimes x) = \deg r + \deg x$.

Now, we have a sequence

$$\mathcal{K}(M)_2 \xrightarrow{\mathrm{d}_2} \ker \mathrm{d}_1 \to H_1(\mathcal{K}(M)) \to 0$$

which is by definition exact in the middle and on the right, so $\ker \mathrm{d}_1$ is generated as an A-module in degree at most $\max\{\deg \operatorname{im} \mathrm{d}_2, \deg(H_1(\mathcal{K}(M)))\}$. Now, the composition

$$\mathcal{K}(M)_2 \xrightarrow{\mathrm{d}_2} \ker \mathrm{d}_1 \xrightarrow{\alpha} R_{>0} \otimes_R M$$

is zero, since it maps $\omega \otimes \underline{g} \otimes x$ to the element

$$r\left(\left(\left(\tau_{0,1}^{\omega}\right)^{-1}\underline{g}\right)_1\right)\cdot r\left(\left(\left(\tau_{0,1}^{\omega}\right)^{-1}\underline{g}\right)_2\right)\otimes x \\ - r\left(\left(\left(\tau_{1,1}^{\omega}\right)^{-1}\underline{g}\right)_1\right)\cdot r\left(\left(\left(\tau_{1,1}^{\omega}\right)^{-1}\underline{g}\right)_2\right)\otimes x$$

which equals zero because $\left(\tau_{0,1}^{\omega}\right)^{-1}\underline{g}$ and $\left(\tau_{1,1}^{\omega}\right)^{-1}\underline{g}$ are equivalent up to the Hurwitz action. In other words, the elements of $\operatorname{im} \mathrm{d}_2 \subset \ker \mathrm{d}_1 \subset \mathcal{K}(M)_1$ are killed by α. Therefore, $\alpha(\ker \mathrm{d}_1)$ is generated as an A-module in degree $\leq \deg(H_1(\mathcal{K}(M)))$.

Now, as α is surjective, this implies $\deg(\ker\beta) \leq \deg(H_1(\mathcal{K}(M)))$ (recall that A is graded trivially). But the exact sequence $0 \to R_{>0} \to R \to A \to 0$ is a projective resolution of $A = R/R_{>0}$; tensoring with M, we get an identification $H_1(M) = \operatorname{Tor}_1^R(A, M) = \ker\beta$. Thus, we have proved (5.18). □

5.5. The proof

Proof of Theorem 5.7. We follow the proof strategy of [EVW16, Thm. 6.1], with an extra focus on the determination of the explicit stable range.

By assumption, $D_R = D_R(U)$ is finite and positive. We know from Remark 3.15 that R is generated in degree one as an algebra over the commutative ring A. From these facts, we conclude that R is A-stabilized by U.

As before, let $M_p = \bigoplus_{n\geq 0} H_p(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; A)$ be the graded R-module corresponding to the p-th homologies of Hurwitz spaces. In order to prove the theorem, we need to show that multiplication by U, $M_p \xrightarrow{U\cdot} M_p$, is an isomorphism in source degree $n > (8D_R + \deg U)p + 7D_R + \deg U$. To see this, we show that

$$\deg(H_q(\mathcal{K}(M_p))) \leq D_R + \deg U + (8D_R + \deg U)p + D_R q \tag{5.19}$$

holds for all $q \geq 0$. Then, the theorem follows from the second statement of Proposition 5.34, considering the cases $q = 0, 1$. We prove (5.19) by induction on p.

For $p = 0$, we have $M_0 = R$. Now, by Lemma 5.32,

$$\deg(H_q(\mathcal{K}(R))) \overset{5.32}{\leq} D_R + \deg U + q \leq D_R + \deg U + D_R q,$$

which implies the assertion.

For the inductive step, suppose that (5.19) holds for $0 \leq p < P$. The final terms of

$\mathcal{K}(M_P)$ are given by

$$\mathcal{K}(M_P)_2 \overset{\mathrm{d}_2}{\to} \mathcal{K}(M_P)_1 \overset{\mathrm{d}_1}{\to} M_P.$$

The n-th graded part of d_2 is a differential $\mathrm{d}\colon E^1_{1P} \to E^1_{0P}$ in the spectral sequence from Corollary 5.19.

The map d_1 is induced by $\omega \otimes g \otimes x \mapsto r(g) \cdot x$. On the other hand, the edge map

$$E^1_{0P} = H_p(E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} (\mathcal{O}_0^{[n,\underline{\xi}]} \times \mathbf{c}^n); A) \to H_p(E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} (|\mathcal{O}^{[n,\underline{\xi}]}| \times \mathbf{c}^n); A)$$

is induced by the geometric realization of vertices, $\mathcal{O}_0^{[n,\underline{\xi}]} \to |\mathcal{O}^{[n,\underline{\xi}]}|$. By the high connectivity of $|\mathcal{O}^{[n,\underline{\xi}]}|$ (cf. the proof of Proposition 5.14), the map

$$H_p(E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} (|\mathcal{O}^{[n,\underline{\xi}]}| \times \mathbf{c}^n); A) \to H_p(E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} \mathbf{c}^n; A) = H_p(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; A),$$

induced by $|\mathcal{O}^{[n,\underline{\xi}]}| \to \mathrm{pt}$, is an isomorphism for $p \leq \lfloor \frac{n}{2} \rfloor - 2$. Hence, the concatenation of the two above maps

$$E^1_{0P} = H_p(E\mathrm{Br}_{n\cdot\underline{\xi}} \times_{\mathrm{Br}_{n\cdot\underline{\xi}}} (\mathcal{O}_0^{[n,\underline{\xi}]} \times \mathbf{c}^n); A) \to H_p(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; A)$$

is induced by $(e, (\omega, \sigma L_0), \underline{g}) \to (e, \underline{g})$. Using the identifications made in (5.3), this becomes

$$(\omega, g_1, (e, (g_2, \ldots, g_n))) \mapsto (e, \underline{g}),$$

which, after applying homology, is just the n-th graded part of the map d_1. Therefore, in the range $p \leq \lfloor \frac{n}{2} \rfloor - 2$, we can identify d_1 with an edge map in the spectral sequence from Corollary 5.19.

Now, E^2_{qp} is given by the n-th graded part of $H_{q+1}(\mathcal{K}(M_p))$. From the inductive hypothesis, for $j > 1$, we obtain $E^2_{j,P+1-j} = 0$ for

$$n > 10D_R + 2\deg U - (7D_R + \deg U)j + (8D_R + \deg U)P.$$

Hence, we have $E^2_{0P} = E^\infty_{0P}$ for $n > -4D_R + (8D_R + \deg U)P$, as there are no nonzero differentials going into or out of E^2_{0P}.

Similarly, for $j > 0$, we have $E^2_{j,P-j} = 0$ for

$$n > 2D_R + \deg U - (7D_R + \deg U)j + (8D_R + \deg U)P.$$

Thus, for $n > -5D_R + (8D_R + \deg U)P$, the only graded piece of $H_P(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; A)$ which does not vanish is E^∞_{0P}.

Combining these results, we see that $E^2_{0P} \cong E^\infty_{0P} \cong H_P(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; A)$ as long as $n > -4D_R + (8D_R + \deg U)P$ (note that this implies $P \leq \lfloor \frac{n}{2} \rfloor - 2$, so that the identifications made above are valid). In particular, the edge map $\mathrm{coker}(d_2) \to M_P$ is an isomorphism in degrees above $-4D_R + (8D_R + \deg U)P$, and so

$$\max\{\deg(H_0(\mathcal{K}(M_P))), \deg(H_1(\mathcal{K}(M_P)))\} \leq -4D_R + (8D_R + \deg U)P. \tag{5.20}$$

Now, we make use of the first statement of Proposition 5.34 and obtain

$$\begin{aligned} \deg(H_q(\mathcal{K}(M_P))) &\overset{(5.20)}{\leq} -4D_R + (8D_R + \deg U)P + D_R q + 5D_R + \deg U \\ &= D_R + \deg U + (8D_R + \deg U)P + D_R q, \end{aligned}$$

which is what we wanted to show. $\square$

5.6. Unmarked covers

The G-action on Hurwitz vectors by simultaneous conjugation induces an action of G on the Hurwitz spaces $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$. The corresponding *Hurwitz space for unmarked covers* is defined as the quotient $\mathcal{H}^c_{G,n\cdot\underline{\xi}} = \mathrm{Hur}^c_{G,n\cdot\underline{\xi}} / G$ (cf. Definition 3.6). Now, in general, G does not act freely on Hurwitz vectors, so the quotient map is not a covering space map. Anyway, for suitable stabilizing elements U, homological stability for Hurwitz spaces descends to spaces of unmarked covers, as we will see in this section's theorem.

The ring R of connected components is generated as an A-module by the elements of $\coprod_{n\geq 0} \mathbf{c}^n / \mathrm{Br}_{n\cdot\underline{\xi}}$. As the Hurwitz action commutes with conjugation, the G-action on Hurwitz vectors gives R (and the modules M_p) the structure of modules over the group ring $A[G]$.

Theorem 5.35. *Let A be a field whose characteristic is either zero or prime to the order of G. Assume that there is a G-invariant element $U \in R$ which induces isomorphisms $H_p(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; A) \cong H_p(\mathrm{Hur}^c_{G,(n+\deg U)\cdot\underline{\xi}}; A)$ for any $p \geq 0$ and $n \geq r(p)$. Then, $H_p(\mathcal{H}^c_{G,n\cdot\underline{\xi}}; A) \cong H_p(\mathcal{H}^c_{G,(n+\deg U)\cdot\underline{\xi}}; A)$ holds in the same range.*

Proof. The proof relies on a fact from equivariant topology: If G is a finite group acting on a space X and A a field of characteristic zero or prime to $|G|$, there are isomorphisms $H_p(X/G; A) \cong H_p(X; A)_G$ for all $p \geq 0$. Here, $H_p(X; A)_G$ denotes the G-coinvariants, i.e., the quotient of $H_p(X; A)$ by the submodule generated by the elements of the form $x - g \cdot x$, for $x \in H_p(X; A)$, $g \in G$. This is the homological version of the *generalized transfer argument*, cf. [Bre72, Thm. 3.2.4]. Thus, we have isomorphisms $H_p(\mathcal{H}^c_{G,n\cdot\underline{\xi}}; A) \xrightarrow{\sim} H_p(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; A)_G$ for all $n, p \geq 0$.

The assumption that U is fixed under the action of G, together with the G-equivariance (cf. also the proof of Proposition 3.12) of the maps $G^{n\cdot\xi} \times G^{\deg U \cdot \xi} \to G^{(n+\deg U)\cdot\xi}$, implies that in the stable range, $H_p(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; A) \cong H_p(\mathrm{Hur}^c_{G,(n+\deg U)\cdot\underline{\xi}}; A)$ is an isomorphism of $A[G]$-modules. Taking G-coinvariants yields the result. $\square$

Remark 5.36. Usual choices for stabilizing elements satisfy the criterion from Theorem 5.35: Both the element U from [EVW16] (cf. Proposition 5.11) and the elements $V(\underline{a})$ from [EVW12] which are introduced in (6.7) are fixed under the conjugation action of G.

6. Application and Further Investigation

This chapter is about the application of the methods and results from Chapter 5 to specific settings in order to obtain concrete homological stability results for Hurwitz spaces. We have already considered two special cases: Purely abelian covers have been treated in Section 5.1. The corresponding Hurwitz spaces are homotopy equivalent to colored configuration spaces, which are known to be homologically stable by Section 2.2.2. The case of non-splitting conjugacy classes is the subject of the result by Ellenberg-Venkatesh-Westerland (Theorem 5.9) which is by now a special case of Theorem 5.7.

In Section 6.1, we consider symmetric covers. The simplest non-abelian case $G = \mathcal{S}_3$ is of special interest. We show that parts of the methods from Chapter 5 can be applied even in cases where we do not investigate a sequence of type $\{\mathrm{Hur}^c_{G,n\cdot\underline{\xi}} \mid n \geq 0\}$. More concretely, we are interested in homological stabilization in the direction of a particular conjugacy class c_0, i.e., we fix a basic Nielsen class and attach branch points from c_0.

Section 6.2 is essentially about the case where all elements of $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$, for all $n \geq 1$, represent connected covers. If this holds, we are able to formulate a homological stability theorem in full generality. The remainder of the chapter is an outlook on further research and open questions.

6.1. Symmetric covers and vertical stabilization

Until now, our investigations involved sequences of type $\{\mathrm{Hur}^c_{G,n\cdot\underline{\xi}} \mid n \geq 0\}$, the *diagonal direction*. It seems as if the homology, and in particular the stable homology of Hurwitz spaces is closely connected to the colored configuration spaces they cover. Indeed, we have already seen an indication in the case of Abelian covers. Theorem 6.14 will be a more general evidence for this. Thence, we might go by by the existing theorems for colored configuration spaces. By Theorem 2.23, there is an isomorphism $H_p(\mathrm{Conf}_{\underline{\mu}};\mathbb{Z}) \to (\mathrm{Conf}_{\underline{\mu}+e_i};\mathbb{Z})$ with stable range $\mu_i \geq 2p$ for any fixed partition $\underline{\mu} \in \mathbb{N}^t$ and the i-th standard unit vector e_i, for $i = 1, \dots, t$. Passing to Hurwitz spaces, this corresponds to the successive addition of branch points of a fixed conjugacy type.

We consider the case where $G = \mathcal{S}_3$ is the symmetric group on three elements. Let $c = (c_2, c_3)$, where $c_2 = \{(12), (13), (23)\}$ is the conjugacy class of transpositions, and $c_3 = \{(123), (132)\}$ the conjugacy class of 3-cycles. We fix an odd number n_2 and consider the sequence $\{\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3)} \mid n_3 \geq 0\}$. The reasons for the consideration of this particular case is made clear in the following. We call this the *vertical direction* (the *horizontal direction* being given by fixing n_3 and increasing n_2).

In this section, elements of the symmetric group appear as entries of Hurwitz vectors. Multiplication of these elements corresponds to the concatenation of loops in D in order to consider the loops' monodromy action on the fiber of a covering map. Thus, it seems natural to perform multiplication in $\mathcal{S}_3$ *from left to right*.

6.1.1. Hurwitz orbits

From Remark 3.4 and Proposition 3.9(i), we know that the number of components of $\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3)}$ is given by the number of Br_{n_2,n_3}-orbits in $c_2^{n_2} \times c_3^{n_3}$ under the Hurwitz action (3.1). In this section, we explicitly determine the number of orbits for any pair $(n_2, n_3) \in \mathbb{N}_0^2$. We do not claim originality to this. In particular, the Hurwitz orbits in D_k^n for arbitrary dihedral groups D_k are determined explicitly in [Sia09].

The easiest case is the *Abelian* case $n_2 = 0$ (cf. also Section 5.5):

Lemma 6.1. *There are exactly $n + 1$ orbits for the Hurwitz action of Br_n on c_3^n.*

Proof. The conjugacy class c_3 generates the alternating group $\mathcal{A}_3 \subset \mathcal{S}_3$ which is abelian. Then, (5.1) implies the assertion. $\square$

Lemma 6.2. *Let $n_2 \geq 1$ and $n_3 \geq 0$, where $(n_2, n_3) \notin \{(1, 0), (2, 0)\}$. Then, the set $\Delta_{n_2,n_3} = \{\partial \underline{g} \mid \underline{g} \in c_2^{n_2} \times c_3^{n_3}, \textit{non-constant}\}$ of possible boundaries is equal to $\mathcal{A}_3$ if n_2 is even and equal to c_2 if n_2 is odd.*

Remark 6.3. For $(n_2, n_3) = (1, 0)$, there are (trivially) no non-constant tuples. In the case $(2, 0)$, there are no non-constant tuples with trivial boundary, so $\Delta_{n_2,n_3} = c_3$.

Proof of Lemma 6.2. We prove the claim by writing down tuples in $c_2^{n_2} \times c_3^{n_3}$ with a prescribed boundary. Note that any element of $\mathcal{A}_3$ can be written as the product of two transpositions. The product of n_3 3-cycles, where $n_3 \geq 2$, may attain every value in $\mathcal{A}_3$ (where we allow constant n_3-tuples). For $x \in \mathcal{A}_3$, let $\underline{g(x)} \in c_3^{n_3}$ be an n_3-tuple with $\partial \underline{g(x)} = x$. Let now x_2, y_2 be different transpositions, $x_3 \in \mathcal{A}_3$ and $y_3 \in c_3$.

- If n_2 is even and $n_3 \geq 2$, $\partial(y_2^{(n_2)}, \underline{g(x_3)}) = x_3$.

- If n_2 is even and $n_3 = 1$, $\partial(\{y_2\}^{n_2-1}, y_2x_3y_3^{-1}, y_3) = x_3$.
- If $n_2 \geq 4$ is even and $n_3 = 0$, let $x_3 = rr'$ be a factorization of x_3 into transpositions, and $s \in c_2$ different from r and r'. Then $\partial(s^{(n_2-2)}, r, r') = x_3$.
- If n_2 is odd and $n_3 \neq 1$, $\partial(x_2^{(n_2)}, \underline{g(\mathrm{id})}) = x_2$.
- If n_2 is odd and $n_3 = 1$, $\partial(y_2^{(n_2)}, y_2x_2) = x_2$. □

Lemma 6.4. *For $n_2 + n_3 \geq 2$ and $n_2 \geq 1$, the group Br_{n_2,n_3} acts transitively on the set of non-constant tuples in $c_2^{n_2} \times c_3^{n_3}$ with prescribed boundary.*

Proof. We prove the following claim by induction on $n = n_2 + n_3 \geq 2$, where $n_2 \geq 1$: If tuples $\underline{g} = (g_1, \ldots, g_n)$ and $\underline{h} = (h_1, \ldots, h_n)$ in $c_2^{n_2} \times c_3^{n_3}$, both non-constant, satisfy $\partial\underline{g} = \partial\underline{h}$, they are equivalent under the Br_{n_2,n_3}-action.

Some more notation: If $X \subset c_2^{n_2} \times c_3^{n_3}$ is Br_{n_2,n_3}-invariant and $h \in \mathcal{S}_3$, we define $(X)_h = \{\underline{g} \in X \mid \partial\underline{g} = h\}$.

The base case $n = 2$ can be verified by direct computation. Applying the generator $\sigma_1 \in \mathrm{Br}_2 \cong \mathbb{Z}$, we obtain two orbits for the action on c_2^2, namely

$$\begin{aligned}(c_2^2)_{(123)} &= \{((12),(13)),((23),(12)),((13),(23))\},\\ (c_2^2)_{(132)} &= \{((13),(12)),((23),(13)),((12),(23))\}.\end{aligned} \tag{6.1}$$

If we apply the generator $\sigma_1^2 \in \mathrm{Br}_{1,1} \cong \mathbb{Z}$, we obtain three orbits in $c_2 \times c_3$, given by

$$\begin{aligned}(c_2 \times c_3)_{(12)} &= \{((13),(132)),((23),(123))\},\\ (c_2 \times c_3)_{(13)} &= \{((12),(123)),((23),(132))\},\\ (c_2 \times c_3)_{(23)} &= \{((12),(132)),((13),(123))\}.\end{aligned} \tag{6.2}$$

Hence, in both cases the orbits are in bijection to the possible boundary values, cf. also Lemma 6.2. Let now $n \geq 3$.

First, consider the case $n_2 \geq 2$. There is a braid $\sigma \in \mathrm{Br}_{n_2,n_3}$ such that $\underline{g'} = \sigma \cdot \underline{g}$ shares its first entry with $\underline{h}$. Indeed, we may assume that the first n_2 entries of $\underline{g}$ are non-constant; if they are constant (this necessarily implies $n_3 > 0$), we apply the braid $\sigma_{n_2}^2$ and obtain a tuple whose first n_2 entries are non-constant, cf. (6.2). Let now $g_k \neq g_1$ be a transposition. The element

$$\gamma = \sigma_{k-1} \cdots \sigma_2\sigma_1\sigma_2^{-1} \cdots \sigma_{k-1}^{-1} \in \mathrm{Br}_{n_2,n_3}$$

replaces the pair (g_1, g_k) by $\sigma_1 \cdot (g_1, g_k)$. Then, (6.1) shows that repeated application of γ yields a tuple with first entry $h_1 \in c_2$.

Now, $(g'_2, \dots, g'_n)$ and $(h_2, \dots, h_n)$ have the same shape and boundary. If we show that we can assume that these $(n-1)$-tuples are non-constant, the induction hypothesis provides a braid $\sigma' \in \mathrm{Br}_{n-1} \subset \mathrm{Br}_n$ such that $\underline{g}' = \underline{h}$. Here, the inclusion $\mathrm{Br}_{n-1} \hookrightarrow \mathrm{Br}_n$ is given by adding a trivial strand on the *left* hand side of a braid. This implies $\underline{g} \sim \underline{g}' \sim \underline{h}$, which we wanted to show.

Let $n > 3$. If, without loss of generality, $(h_2, \dots, h_n) \in c_2^{n_2 - 1}$ is constant, application of $\sigma_1 \sigma_2^2 \sigma_1$ to $\underline{h}$ results in a new tuple $\underline{h}'$ with the same first entry and $(h'_2, \dots, h'_n)$ non-constant, since for x, y transpositions other than h_1,

$$\sigma_1 \sigma_2^2 \sigma_1 \cdot (h_1, x, x) = (h_1, y, y). \tag{6.3}$$

For $n = 3$, the tuple (h_1, x, x) is by (6.3) Br_3-equivalent to (h_1, y, y), and there are no other triples in c_2^3 with both first entry and boundary equal to h_1. Thus, $\underline{g}'$ and $\underline{h}$ are equivalent in this case.

Secondly, let $n_2 = 1$. We show that we can adjust $\underline{g}$ to a tuple $\underline{g}'$ with *last* entry h_n. Indeed, we may by (6.2) assume that both kinds of 3-cycles appear in the last n_3 entries of $\underline{g}$. As the Hurwitz action on 3-cycles is just permutation, we may pull the desired entry to the end of the n-tuple $\underline{g}$.

In this case, $(g'_1, \dots, g'_{n-1})$ and $(h_1, \dots, h_{n-1})$ are, as elements of $c_2 \times c_3^{n-2}$, non-constant. Therefore, we may apply the induction hypothesis as above, where this time the inclusion $\mathrm{Br}_{n-1} \hookrightarrow \mathrm{Br}_n$ is the usual one. $\square$

We may now combine the lemmas and remarks from Section 4.16. Note that for $n_3 = 0$, there are exactly three constant tuples in any degree n_2.

$$b_0(\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3)}) = \begin{cases} n_3 + 1, & n_2 = 0 \\ 3, & (n_2, n_3) = (1, 0) \\ 5, & (n_2, n_3) = (2, 0) \\ 6, & n_2 \geq 3, n_3 = 0 \\ 3, & n_2, n_3 \geq 1. \end{cases} \tag{6.4}$$

Example 6.5. Applying the above results, we see that for fixed $n_2, n_3 \geq 1$, any Hurwitz vector $U \in (c_2^{n_2} \times c_3^{n_3})^k$ with $\partial U = 1$ induces isomorphisms $R^{\mathbb{Z},c}_{\mathcal{S}_3,(n_2,n_3)} \to R^{\mathbb{Z},c}_{\mathcal{S}_3,(n_2,n_3)}$ in all degrees $q \geq 1$. Thus, we have $D_R(U) = \deg U = k$. For all $p \geq 0$, Theorem 5.7 yields $H_p(\mathrm{Hur}^c_{\mathcal{S}_3,m\cdot(n_2,n_3)}; \mathbb{Z}) \cong H_p(\mathrm{Hur}^c_{\mathcal{S}_3,(m+k)\cdot(n_2,n_3)}; \mathbb{Z})$ with stable range $m > 9kp + 8k$.

6.1.2. The reduced ring of connected components

Fix an odd number $n_2 \geq 1$. We consider the sequence $\{\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3)} \mid n_3 \geq 0\}$ and the $\mathbb{Z}$-module $M_p^{n_2} = \bigoplus_{n_3 \geq 0} H_p(\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3)};\mathbb{Z})$. Proposition 3.12 yields a unique homomorphism

$$H_0(\mathrm{Hur}^c_{\mathcal{S}_3,(0,j)};\mathbb{Z}) \times H_p(\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3)};\mathbb{Z}) \to H_p(\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3+j)};\mathbb{Z})$$

just like in (3.5), for all $j, n_2, n_3, p \in \mathbb{N}_0$. Therefore, M_p is a graded module over the graded ring $R = R^{\mathbb{Z},c_3}_{\mathcal{S}_3,1} = \bigoplus_{n_3 \geq 0} H_0\left(\mathrm{Hur}^{c_3}_{\mathcal{S}_3,n_3};\mathbb{Z}\right)$, (cf. Definition 3.14), where the grading is in the n_3-variable.

As a consequence of Remark 3.15 and Lemma 6.1, the ring R is isomorphic to the polynomial algebra $\mathbb{Z}[x,y]$, where the powers of x and y denote the number of entries in a Hurwitz vector which are equal to (123) and (132), respectively. Thus, R *cannot* be $\mathbb{Z}$-stabilized by *any* element $U \in R$ since there is no chance of an isomorphism from degree n to degree $n+k$, for any $k \in \mathbb{N}$. Therefore, it makes sense to consider a reduced version of R with a constant number of copies of $\mathbb{Z}$ in degree n_3, for n_3 sufficiently large:

Definition 6.6. The *reduced ring of connected components* is the graded ring (grading induced by the grading on R) defined by $\tilde{R} = R/\mathfrak{a}$, where $\mathfrak{a}$ is the homogeneous principal ideal generated by $x^3 - y^3$.

Lemma 6.7. *The ring $\tilde{R}$ is $\mathbb{Z}$-stabilized by $U = x^3\mathfrak{a}$, and we have $D_{\tilde{R}}(U) = 4$.*

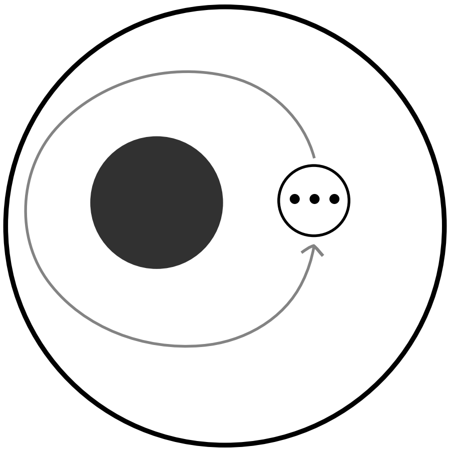
Proof. As a $\mathbb{Z}$-module, the n-th graded part of $\tilde{R} \cong \mathbb{Z}[x,y]/(x^3 - y^3)$, $n \geq 2$, is the free $\mathbb{Z}$-module spanned by $x^n\mathfrak{a}, x^{n-1}y\mathfrak{a}$, and $x^{n-2}y^2\mathfrak{a}$. Hence, multiplication by $x^3\mathfrak{a}$ is an isomorphism in target degree $n \geq 5$, and $\deg(\tilde{R}/U\tilde{R}) = 4$; thus, $D_{\tilde{R}}(U) = 4$. $\square$

Lemma 6.8. *Let $n_2 \in \mathbb{N}$ be odd and $n_3 \in \mathbb{N}_0$. Then, there is a braid $\sigma \in \mathrm{Br}_{n_2,n_3+3}$ which centralizes the subgroup $\mathrm{Br}_{n_2,n_3} \subset \mathrm{Br}_{n_2,n_3+3}$ (new strands added from the right), and which satisfies $\sigma \cdot (\underline{g}, (123)^{(3)}) = (\underline{g}, (132)^{(3)})$ for all $\underline{g} \in c_2^{n_2} \times c_3^{n_3}$.*

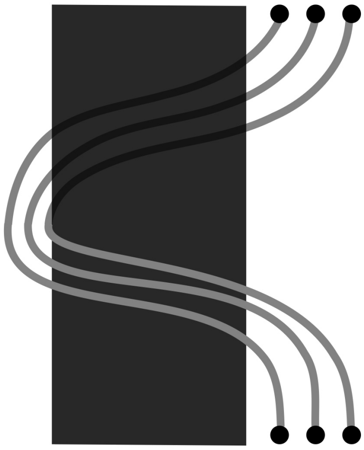
Proof. The result of application of the braid

$$\sigma = \left(\prod_{i=1}^{3}\prod_{j=1}^{n_2+n_3} \sigma_{n_2+n_3+i-j}\right)\left(\prod_{k=1}^{3}\prod_{l=1}^{n_2+n_3} \sigma_{l-k+3}\right)$$

(cf. Figure 6.1a) to $(\underline{g}, (123)^{(3)})$ is conjugation of the last three entries (123) by $\partial\underline{g}$. Now, $\partial\underline{g}$ is a transposition due to the assumption that n_2 is odd. Thus, we have $\sigma \cdot (\underline{g}, (123)^{(3)}) = (\underline{g}, (132)^{(3)})$.



(a) Mapping class action of σ.



(b) The braid σ^{-1}.

Figure 6.1.: The braid $\sigma \in \mathrm{Br}_{n_2,n_3+3}$ stabilizes Br_{n_2,n_3}.

It remains to argue that σ centralizes Br_{n_2,n_3}. This is shown in Figure 6.1b, where σ^{-1} is sketched as an Artin braid. Any strands inside the black box (that is, in the subgroup Br_{n_2,n_3}) are unaffected by the three remaining strands winding around them, so any element of Br_{n_2,n_3} is centralized by σ, cf. also [FRZ96]. □

Proposition 6.9. *For any odd number n_2, $M_p^{n_2}$ is a graded left $\tilde{R}$-module.*

Proof. Since we already know that $M_p^{n_2}$ is a graded left R-module, it remains to show that the elements $U = x^3 \in R$ and $U' = y^3 \in R$ induce the same map

$$H_p(\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3)};\mathbb{Z}) \to H_p(\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3+3)};\mathbb{Z}).$$

We proceed by proving that the corresponding maps on Hurwitz spaces are homotopic. Let $B(\mathrm{Br}_{n_2,n_3})_{\underline{g}}$ be the connected component of $\mathrm{Hur}^{c_3}_{\mathcal{S}_3,3}$ corresponding to the Hurwitz vector $\underline{g} \in \mathrm{Br}_{n_2,n_3}$ (cf. Proposition 3.9(ii)).

In the proof of Proposition 3.12, we constructed the multiplication maps from the corresponding maps of stabilizer subgroups of the Artin braid group. In particular, on the level of connected components, U and U' induce maps of classifying spaces of stabilizer subgroups:

$$\begin{aligned} U_*\colon\ & B(\mathrm{Br}_{n_2,n_3})_{\underline{g}} \to B(\mathrm{Br}_{n_2,n_3+3})_{(\underline{g},(123)^{(3)})}, \\ U'_*\colon\ & B(\mathrm{Br}_{n_2,n_3})_{\underline{g}} \to B(\mathrm{Br}_{n_2,n_3+3})_{(\underline{g},(132)^{(3)})}. \end{aligned}$$

These maps are induced by the restriction of the inclusion $\kappa\colon \mathrm{Br}_{n_2,n_3} \hookrightarrow \mathrm{Br}_{n_2,n_3+3}$

defined by adding three trivial strands on the right hand side of a braid. By Lemma 6.4, the target of both maps is the same up to the choice of a base point, as both $((123)^{(3)})$ and $((132)^{(3)})$ have trivial boundary.

Changing base points from $(\underline{g}, (132)^{(3)})$ to $(\underline{g}, (123)^{(3)})$ corresponds to conjugation by a braid $\sigma \in \mathrm{Br}_{n_2,n_3+3}$ such that

$$\sigma \cdot (\underline{g}, (123)^{(3)}) = (\underline{g}, (132)^{(3)}). \tag{6.5}$$

Applying the theory of classifying spaces, this gives the following criterion: U_* and U'_* are freely homotopic if and only if there is a $\sigma \in \mathrm{Br}_{n_2,n_3+3}$ such that the map

$$\begin{aligned} (\mathrm{Br}_{n_2,n_3})_{\underline{g}} &\to (\mathrm{Br}_{n_2,n_3+3})_{(\underline{g},(123)^{(3)})} \\ \gamma &\mapsto \sigma^{-1}\kappa(\gamma)\sigma \end{aligned}$$

is conjugate in $(\mathrm{Br}_{n_2,n_3+3})_{(\underline{g},(123)^{(3)})}$ to the inclusion $\gamma \mapsto \kappa(\gamma)$.

Since the braid σ from Lemma 6.8 centralizes $\mathrm{Br}_{n_2,n_3} \cong \operatorname{im}\kappa$, the map $\gamma \mapsto \sigma^{-1}\gamma\sigma$ is the identity for this choice of σ. This implies the proposition. $\square$

6.1.3. The spectral sequence and homological stability

The space $\mathrm{Hur}^c_{G,n_2,n_3}$ covers $B\mathrm{Br}_{n_2,n_3}$. In order to obtain a spectral sequence which converges to the homology of these Hurwitz spaces, we wish to find a simplicial complex with a Br_{n_2,n_3}-action and suitable stabilizers. In the present case, we can cling to the arc complex $\mathcal{A} = \mathcal{P}^1_{n_3}(D_{n_2})$ on an n_2-punctured disk D_{n_2}.

Applying Theorem 4.11, we see that the geometric realization of this complex is $(n_3 - 2)$-connected.

Lemma 6.10. *For all $q \geq 0$, the colored braid group Br_{n_2,n_3} acts transitively on the set $\mathcal{A}_q$ of q-simplices. The stabilizer of a q-simplex is isomorphic to $\mathrm{Br}_{n_2,n_3-q-1}$.*

Proof. The group $\mathrm{Br}_{n_2+n_3}$ is isomorphic to the mapping class group of a disk with $(n_2 + n_3)$ punctures (or marked points). Then, Br_{n_2,n_3} can be identified with the subgroup which leaves a set of n_2 punctures and a set of n_3 marked points invariant. In analogy to the proof of Lemma 4.14, it follows that Br_{n_2,n_3} acts on $\mathcal{A}_q$.

The remainder of the proof is a direct application of the methods from the proofs of Lemma 4.16 and Lemma 4.18: Since any vertex of $\mathcal{A}$ consists of only one arc, there is only one IC-sequence and the action of Br_{n_2,n_3} on $\mathcal{A}_q$ is transitive for all q. For a q-simplex σ with representative union of arcs Σ, we show just as in 4.18 that the map $\mathrm{Map}(D_{n_2,n_3-q-1}, \Sigma) \to (\mathrm{Map}(D_{n_2,n_3}))_\sigma$ is an isomorphism. $\square$

By Lemma 6.10, we may identify $\mathcal{A}_q$ with $\mathrm{Br}_{n_2,n_3}/L_q \cong \mathrm{Br}_{n_2,n_3}/\mathrm{Br}_{n_2,n_3-q-1}$, where L_q stabilizes the q-simplex consisting of the straight lines from $*$ to the first $q+1$ of the n_3 marked points. We apply the methods from either Section 4.2 (in a much simpler fashion) or Section 5 of [EVW16]: We identify $\partial_i\colon \mathcal{A}_q \to \mathcal{A}_{q-1}$ with the map $\sigma L_q \mapsto \sigma\tau_{q,i}L_{q-1}$, where $\tau_{q,i} = \sigma_{n_2+i+1}\cdot\ldots\cdot\sigma_{n_2+q}$.

Proposition 6.11. *There exists a homological spectral sequence with*

$$E^1_{qp} = \mathbb{Z}\langle c_3^{q+1}\rangle \otimes_{\mathbb{Z}} H_p(\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3-q-1)};\mathbb{Z})$$

for $p,q \geq 0$ *and target* $H_{p+q}(\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3)};\mathbb{Z})$ *in degrees* $p+q \leq n_3-2$.

Proof. This is by now analogous to Section 5.3. The semi-simplicial structure on the (n_3-1)-dimensional complex $\mathcal{A}$ carries over to $E\mathrm{Br}_{n_2,n_3} \times_{\mathrm{Br}_{n_2,n_3}} (\mathcal{A}\times c_3^{n_3})$, cf. also Lemma 5.12. Now, the target of the corresponding spectral sequence

$$\begin{aligned} E^1_{qp} &= H_p(E\mathrm{Br}_{n_2,n_3}\times_{\mathrm{Br}_{n_2,n_3}}(\mathcal{A}_q\times c_2^{n_2}\times c_3^{n_3});\mathbb{Z}) \\ &\Longrightarrow H_{p+q}(E\mathrm{Br}_{n_2,n_3}\times_{\mathrm{Br}_{n_2,n_2}}(|\mathcal{A}|\times c_2^{n_2}\times c_3^{n_3});\mathbb{Z}) \end{aligned}$$

is isomorphic to $H_{p+q}(\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3)};\mathbb{Z})$ for $p+q\leq n_3-2$ (cf. also Proposition 5.14). Furthermore, we obtain

$$\begin{aligned} E^1_{qp} &= H_p(E\mathrm{Br}_{n_2,n_3}\times_{\mathrm{Br}_{n_2,n_3}}(\mathcal{A}_q\times c_2^{n_2}\times c_3^{n_3});\mathbb{Z}) \\ &\cong H_p(E\mathrm{Br}_{n_2,n_3}\times_{\mathrm{Br}_{n_2,n_3}}(\mathrm{Br}_{n_2,n_3}/L_q\times c_2^{n_2}\times c_3^{n_3});\mathbb{Z}) \\ &\cong H_p(c_3^{q+1}\times(E\mathrm{Br}_{n_2,n_3}\times_{L_q}(c_2^{n_2}\times c_3^{n_3-q-1}));\mathbb{Z}) \\ &\cong \mathbb{Z}\langle c_3^{q+1}\rangle\otimes_{\mathbb{Z}} H_p(\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3-q-1)};\mathbb{Z}) \end{aligned}$$

just as in (5.3) and (5.4). □

As in Definition 5.16, we obtain the *$\mathcal{K}$-complex* of a graded left R-module M as the complex $\mathcal{K}(M)$ with q-th graded part $\mathcal{K}(M)_q = \mathbb{Z}\langle c_3^q\rangle\otimes_{\mathbb{Z}} M(q)$ and differential

$$\begin{aligned} \mathrm{d}_{q+1}\colon\ &\mathcal{K}(M)_{q+1}\to\mathcal{K}(M)_q \\ (g_0,\ldots,g_q)\otimes x &\mapsto \sum_{i=0}^{q}(-1)^i\,[(g_0,\ldots,\hat{g}_i,\ldots,g_q)\otimes r(g_i)\cdot x] \end{aligned}$$

for $q=0,1,\ldots,$. From Proposition 6.11 and an application of the methods from Lemma 5.15, we obtain that the n_3-th graded part of $\mathcal{K}(M_p^{n_2})$ is isomorphic to the q-th row of the spectral sequence, cf. also [EVW16, Lemma 5.4] and Corollary 5.19.

Of course, we are concerned with the special case where M is a graded left module over the reduced ring $\tilde{R}$. Note that, due to the analogous definition of $\mathcal{K}$-complexes and since $\tilde{R}$ is $\mathbb{Z}$-stabilized by $U = x^3$ with $D_{\tilde{R}}(U) = 4$ by Lemma 6.7, the statements from Lemma 5.20 and the whole Section 5.4 are still valid in the current case.

After a final preparatory lemma, we may proceed directly to the proof of a homological stability statement for Hurwitz spaces of $\mathcal{S}_3$-covers.

Lemma 6.12. *For all $q \geq 0$, $\deg(H_q(\mathcal{K}(M_0^{n_2}))) \leq \deg(H_q(\mathcal{K}(\tilde{R}))) + q + 2$.*

Proof. We show the existence of a graded homomorphism $\iota\colon \tilde{R} \to M_0^{n_2}$ of $\tilde{R}$-modules which is an isomorphism in degrees $n_3 \geq 2$. Such a map is automatically compatible with the differentials of the $\mathcal{K}$-complex, i.e., the diagram

$$\begin{tikzcd} \mathcal{K}(\tilde{R})_{q+1} \arrow[r, "\mathrm{d}_{q+1}"] \arrow[d, "\mathcal{K}(\iota)"] & \mathcal{K}(\tilde{R})_q \arrow[d, "\mathcal{K}(\iota)"] \\ \mathcal{K}(M_0^{n_2})_{q+1} \arrow[r, "\mathrm{d}_{q+1}"] & \mathcal{K}(M_0^{n_2})_q \end{tikzcd}$$

commutes, where $\mathcal{K}(\iota)$ is defined by linear extension of $\underline{g} \otimes x \mapsto \underline{g} \otimes \iota(x)$. By definition of the terms of the $\mathcal{K}$-complexes as finite direct sums of summands $\tilde{R}$ and $M_0^{n_2}$, respectively, this implies that $\mathcal{K}(\iota)$ is an isomorphism in degrees $\geq q+2$, which proves the lemma.

Consider Remark 3.15: An application of the combinatorial construction to $\tilde{R}$ and $M_0^{n_2}$, using Lemmas 6.2 and 6.4, yields the following descriptions:

- $\tilde{R}$ is the $\mathbb{Z}$-algebra over the monoid given by the set of pairs

  $$\{(0, \mathrm{id}), (1, (123)), (1, (132))\} \cup \mathbb{N}_{\geq 2} \times \mathcal{A}_3 \subset \mathbb{N}_0 \times \mathcal{A}_3,$$

  where the operation is addition in the first entry (the *degree*) and multiplication in $\mathcal{A}_3$ in the second entry (the *boundary*).

- In degree > 0, $M_0^{n_2}$ is the free $\mathbb{Z}$-module over the set of pairs $\mathbb{N}_{\geq 1} \times c_2$. Degree zero is generated by $c_2^{n_2} / \mathrm{Br}_{n_2}$ – here, the boundary map $c_2^{n_2} / \mathrm{Br}_{n_2} \to \{(0, g) \mid g \in c_2\}$ is always surjective, but injective only for $n_2 = 1$ (cf. 6.4). Addition of the degree and multiplication of the boundary gives the set

  $$(\{0\} \times c_2^{n_2} / \mathrm{Br}_{n_2}) \cup (\mathbb{N}_{\geq 1} \times c_2)$$

  the structure of a left module over the monoid which generates $\tilde{R}$.

Thus, $\iota_0\colon (n_3, g) \mapsto (n_3, g \cdot (12))$ induces a map $\iota\colon \tilde{R} \to M_0^{n_2}$: In degree zero of $M_0^{n_2}$, we fix an arbitrary summand with boundary monodromy (12) as the target of the map. Then, ι satisfies the desired properties: It is a homomorphism of $\tilde{R}$-modules by associativity of the multiplication in $\mathcal{S}_3$ and an isomorphism in degree ≥ 2 as ι_0 is bijective on the second factor for any fixed $n_3 \geq 2$. $\square$

Theorem 6.13. *Let $n_2 \geq 1$ be a fixed odd integer. Then for any $p \geq 0$, there is an isomorphism $H_p(\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3)};\mathbb{Z}) \cong H_p(\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3+1)};\mathbb{Z})$ whenever $n > 35p + 31$.*

Proof. Showing that $U = x^3 \in \tilde{R}$ yields an isomorphism

$$H_p(\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3)};\mathbb{Z}) \cong H_p(\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3+3)};\mathbb{Z}) \tag{6.6}$$

for $n > 35p + 31$ works essentially identically to the proof of Theorem 5.7, since $\tilde{R}$ is $\mathbb{Z}$-stabilized by U (cf. Lemma 6.7). A difference hides in the base case of the induction, where $M_0^{n_2} \neq \tilde{R}$ in this case. By Lemma 6.12 and Lemma 5.32, we have

$$\begin{aligned}\deg(H_q(\mathcal{K}(M_0^{n_2}))) &\overset{6.12}{\leq} \deg(H_q(\mathcal{K}(\tilde{R}))) + q + 2 \\ &\overset{5.32}{\leq} D_{\tilde{R}}(U) + \deg U + 2q + 2 \\ &\leq D_{\tilde{R}}(U) + \deg U + D_{\tilde{R}}(U)q\end{aligned}$$

for $q \geq 1$, as $D_{\tilde{R}}(U) = 4$ by Lemma 6.7.

It remains to check the case $q = 0$. In degree $n_3 \geq 2$, the differential d_1 is the map

$$\begin{aligned}\mathrm{d}_1\colon \mathbb{Z}\langle c_3\rangle \otimes_{\mathbb{Z}} H_0(\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3-1)};\mathbb{Z}) &\to H_0(\mathrm{Hur}^c_{\mathcal{S}_3,(n_2,n_3)};\mathbb{Z}) \\ g \otimes x &\mapsto r(g) \cdot x,\end{aligned}$$

which is surjective. This follows from the combinatorial description in the proof of Lemma 6.12. Hence, $\deg(H_0(\mathcal{K}(M_0^{n_2}))) \leq 1 < D_{\tilde{R}}(U) + \deg U$.

The remainder of the reasoning follows the proof of Theorem 5.7. The insertion of $D_{\tilde{R}}(U) = 4$ and $\deg U = 3$ yields (6.6).

Now, multiplication by $U = x^3$ factors as triple multiplication by $U' = x$, which by the above reasoning must be injective for $n > 35p + 31$, and thus, in fact, an isomorphism. $\square$

6.2. Stable homology and connected covers

We return to a more general setting. As before, let G be a finite group, $c = (c_1, \dots, c_t)$ a list of t distinct conjugacy classes in G, and $\underline{\xi} \in \mathbb{N}^t$. We write $\mathbf{c} = c_1^{\xi_1} \times \dots \times c_t^{\xi_t}$.

6.2.1. Stable homology

In Section 5.1, we noticed that in particular settings, Hurwitz spaces are homotopic to (unions of) colored configuration spaces. By (5.2), the stable integral homology of the sequence $\{\mathrm{Hur}^c_{G,n\cdot\underline{\xi}} \mid n \geq 0\}$ is that of the sequence $\{\mathrm{Conf}_{n\cdot\underline{\xi}} \mid n \geq 0\}$ in the purely Abelian case. This evokes the following question:

> Assume that the sequence $\{\mathrm{Hur}^c_{G,n\cdot\underline{\xi}} \mid n \geq 0\}$ is homologically stable with A-coefficients. What is the stable homology? Is it connected to the stable homology of $\{\mathrm{Conf}_{n\cdot\underline{\xi}} \mid n \geq 0\}$?

For spaces of connected covers, this question was answered by Ellenberg, Venkatesh, and Westerland in the preprint [EVW12]. Due to a false reasoning in the application of results from their earlier article [EVW16] in order to obtain further arithmetic consequences, this preprint was withdrawn from the arXiv in 2013. Fortunately, the statement needed here remains untouched. We refer to Ellenberg's blog post [Ell13] for an explanation of the mistakes and a clarification which results are still correct.

For $\underline{a} = (a_1, \dots, a_t) \in \mathbb{N}^t$, we define the Hurwitz vector

$$V = V(\underline{a}) = \prod_{i=1}^{t} \prod_{g \in c_i} \left(g^{(a_i \operatorname{ord}(g))} \right), \tag{6.7}$$

where the product operation means concatenation of tuples. The element V singles out a connected component of $\mathrm{Hur}^c_{G,m}$, for a suitable $m \in \mathbb{N}$ (cf. Remark 3.15). Now, if there is an $n \in \mathbb{N}$ such that we have $\sum_{g \in c_i} a_i \operatorname{ord}(g) = n\xi_i$ for all $i = 1, \dots, t$, we have $V \in \mathbf{c}^n$ up to the action of $\mathrm{Br}_{n\cdot\underline{\xi}}$. Hence, in this case, $V \in R = R^{A,c}_{G,\underline{\xi}}$, where we used Notation 3.16. Finally, we have $\partial V = \mathrm{id}$, so V is central by Lemma 3.17.

Note that a suitable $\underline{a}$ always exists. Indeed, for all $i = 1, \dots, t$, choose

$$a_i = \xi_i \prod_{j \neq i} \operatorname{ord}(c_j)|c_j|,$$

where $\operatorname{ord}(c_j)$ denotes the order of the elements in c_j, and $|c_j|$ its cardinality. For the rest of Section 6.2, assume that $\underline{a}$ is chosen such that $V \in R$ holds (with suitable coefficients A for the respective situations).

Theorem 6.14 (Ellenberg–Venkatesh–Westerland, [EVW12, Cor. 5.8.2]) *Fix $p \geq 0$, and suppose that the element V from (6.7) induces an isomorphism $H_p(\mathrm{CHur}^c_{G,n\cdot\underline{\xi}};\mathbb{Q}) \xrightarrow{\sim} H_p(\mathrm{CHur}^c_{G,(n+\deg V)\cdot\underline{\xi}};\mathbb{Q})$ for $n \geq r(p)$. Then for any connected component X of $\mathrm{CHur}^c_{G,n\cdot\underline{\xi}}$, the branch point map $X \to \mathrm{Conf}_{n\cdot\underline{\xi}}$ induces an isomorphism $H_p(X;\mathbb{Q}) \xrightarrow{\sim} H_p(\mathrm{Conf}_{n\cdot\underline{\xi}};\mathbb{Q})$ whenever $n \geq r(p)$.*

Example 6.15. Let $G = \mathcal{S}_3$ be the symmetric group on three elements. For $c = (c_2, c_3)$ as in Section 6.1, we consider the case $\underline{\xi} = (1,1)$. As $\mathcal{S}_3$ is generated by any pair in $\mathbf{c} = c_2 \times c_3$, we have $\mathrm{Hur}^c_{\mathcal{S}_3,n\cdot\underline{\xi}} = \mathrm{CHur}^c_{\mathcal{S}_3,n\cdot\underline{\xi}}$ for any $n \geq 0$. Furthermore, from the results of Section 6.1.1, we see that the Br_{12}-orbit of

$$V(1,1) = ((12),(12),(13),(13),(23),(23),(123),(123),(123),(132),(132),(132))$$

defines an element V which $\mathbb{Z}$-stabilizes the ring R. Now, $\deg V = 6$ and $D_R(V) = 6$, so in the range $n > 54p + 48$, we have

$$H_p(\mathrm{CHur}^c_{\mathcal{S}_3,n\cdot\underline{\xi}};\mathbb{Z}) \cong H_p(\mathrm{CHur}^c_{\mathcal{S}_3,(n+6)\cdot\underline{\xi}};\mathbb{Z})$$

by Theorem 5.7. Combining this with Theorem 6.14 and Proposition 2.24 yields

$$\begin{aligned} H_p(\mathrm{CHur}^c_{\mathcal{S}_3,n\cdot\underline{\xi}};\mathbb{Q}) &\overset{6.14}{\cong} \mathbb{Q}^3 \otimes_{\mathbb{Q}} H_p(\mathrm{Conf}_{n\cdot\underline{\xi}};\mathbb{Q}) \\ &\overset{2.24}{\cong} \mathbb{Q}^3 \otimes_{\mathbb{Q}} H_p(\mathrm{Conf}_{(n+1)\cdot\underline{\xi}};\mathbb{Q}) \\ &\overset{6.14}{\cong} H_p(\mathrm{CHur}^c_{\mathcal{S}_3,(n+1)\cdot\underline{\xi}};\mathbb{Q}) \end{aligned}$$

in the same range, where we used that $H_0(\mathrm{CHur}^c_{\mathcal{S}_3,(n+1)\cdot\underline{\xi}};\mathbb{Q}) \cong \mathbb{Q}^3$ by Lemma 6.2 and Lemma 6.4.

6.2.2. Connected covers and invariable generation

In this section, we generalize the strategy applied in Example 6.15. Theorem 5.7 lists requirements for homological stability of the sequence $\{\mathrm{Hur}^c_{G,n\cdot\underline{\xi}} \mid n \geq 0\}$, while Theorem 6.14 is about the spaces $\mathrm{CHur}^c_{G,n\cdot\underline{\xi}}$ *of connected* covers. In order to make the two theorems compatible, the condition

$$\mathrm{Hur}^c_{G,n\cdot\underline{\xi}} = \mathrm{CHur}^c_{G,n\cdot\underline{\xi}} \text{ for all } n \geq 1 \tag{6.8}$$

must be satisfied. By Definition 3.10, condition (6.8) is equivalent to $\mathbf{c} = \mathbf{c} \cap G^{\xi}_{\mathrm{gen}}$. This property is well-examined; we quote some notable results below.

Definition 6.16. We say that G is *invariably generated* by a tuple $c = (c_1, \ldots, c_t)$ of distinct conjugacy classes in G if for all choices of elements $g_i \in c_i$, $i = 1, \ldots, t$, the group is generated by $g_1, \ldots, g_t$. In this case, we call c an *invariable generation system* for G.

The following theorem is due to Camille Jordan. Depending on its interpretation as a theorem in number theory, topology, or group theory, it takes various forms – for an overview, we refer to [Ser03]. Here, we state the theorem in terms of the invariable generation property. In particular, it tells us that any group possesses an invariable generation system.

Theorem 6.17 (Jordan, [Jor72])
If $c = (c_1, \ldots, c_t)$ is a list of all conjugacy classes in G, it invariably generates G.

The next theorem is not directly relevant for the following discussion; we include it to get an idea of the length of invariable generation systems.

Theorem 6.18. *We have the following results for the invariable generation of finite groups G:*
(Kantor–Lubotzky–Shalev, [KLS11])

(i) G is invariably generated by at most $t = \log_2 |G|$ conjugacy classes.

(ii) If G is a non-abelian simple group, it is invariably generated by $t = 2$ conjugacy classes.

(Detomi–Lucchini, [DL15])

(iii) If G is a subgroup of the symmetric group $\mathcal{S}_d$, it is invariably generated by at most $t = \lfloor \frac{d}{2} \rfloor$ conjugacy classes if $d > 3$, and by at most $t = 2$ conjugacy classes for $d = 3$.

The following proposition is a slight adaptation of a standard result about Hurwitz action orbits. In essence, it states that the stabilization of the number of connected components of $\mathrm{CHur}^c_{G,n\cdot\underline{\xi}}$ recognized in the case $G = \mathcal{S}_3$ in Lemma 6.4 follows a more general pattern.

Proposition 6.19. *If c invariably generates G, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, concatenation with any $g \in \mathbf{c}$ yields a bijection*

$$\mathbf{c}^n \,/\, \mathrm{Br}_{n\cdot\underline{\xi}} \xleftrightarrow{1:1} \mathbf{c}^{n+1} \,/\, \mathrm{Br}_{(n+1)\cdot\underline{\xi}},$$

where the braid action is the Hurwitz action. Thus, any Hurwitz vector $U \in R = R^{\mathbb{Z},c}_{G,\underline{\xi}}$ induces an isomorphism $H_0(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}};\mathbb{Z}) \cong H_0(\mathrm{Hur}^c_{G,(n+\deg U)\cdot\underline{\xi}};\mathbb{Z})$ for all $n \geq N$. In particular, $D_R(U)$ is finite for any vector $U \in R$ with $\partial U = 1$.

Proof. The first part of the proposition is usually stated for a single conjugacy class, cf. [MM99, Lemma 6.9(c)]. The main part of our proof follows [EVW16, Prop. 3.4], where the statement (in a slightly different form) is proved for $t = 1$. The last two statements are direct consequences of the first one, applying Proposition 3.9(i) and Lemma 3.17.

Let $\underline{h} \in \mathbf{c}^{n+1}$. We need to show that for n sufficiently large, there is a tuple $\underline{h}' \in \mathbf{c}^n$ such that $\underline{h}$ is equivalent under the $\mathrm{Br}_{n\cdot\underline{\xi}}$-action to $(g, \underline{h}')$. This shows that the maps $\mathbf{c}^n / \mathrm{Br}_{n\cdot\underline{\xi}} \to \mathbf{c}^{n+1} / \mathrm{Br}_{(n+1)\cdot\underline{\xi}}$ given by concatenation with $g = (g_1, \dots, g_\xi)$ are surjective for $n \gg 0$; since the involved sets are finite, it follows that these maps are eventually bijective.

In the following, we work with the full $\mathrm{Br}_{n\xi}$-action. If we construct a tuple $(g, \underline{h}'')$ which is equivalent under the $\mathrm{Br}_{n\xi}$-action to $\underline{h}$, there is another braid which transforms $\underline{h}''$ to an element of $\mathbf{c}^n$, since the Hurwitz action permutes conjugacy types. Thus, it suffices to show that we can realize any $g_0 \in G$ as the first entry of a tuple which is $\mathrm{Br}_{n\xi}$-equivalent to $\underline{h}$; the claim follows by successive application of this property.

Assume $g_0 \in c_1$. As $n \gg 0$, there exists an element $g_0' \in c_1$ that appears at least $d + 1 = \mathrm{ord}(g_0') + 1$ times in $\underline{h}$. We may use the Hurwitz action to pull d of these elements to the front of $\underline{h}$, resulting in a new tuple $({g_0'}^{(d)}, \tilde{h}_1, \dots, \tilde{h}_{n\xi-d})$. By the invariable generation property, the elements $\tilde{h}_1, \dots, \tilde{h}_{n\xi-d}$ generate G (note that g_0' appears at least once in these last $n\xi - d$ entries).

Now for all $i = 1, \dots, n\xi - d$, there is a braid $\sigma_i \in \mathrm{Br}_{n\xi}$ which satisfies

$$\sigma_i \cdot \left({g_0'}^{(d)}, \tilde{h}_1, \dots, \tilde{h}_{n\xi-d}\right) = \left(\left(\tilde{h}_i g_0' \tilde{h}_i^{-1}\right)^{(d)}, \tilde{h}_1, \dots, \tilde{h}_{n\xi-d}\right).$$

It is given by

$$\sigma_i = \alpha_i^{-1}\left(\sigma_{d+i-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}\cdots\sigma_{d+i-1}\right)\alpha_i,$$

where α_i pulls the d-tuple $({g_0'}^{(d)})$ in front of $\tilde{h}_i$, which works since $\partial({g_0'}^{(d)})$ is trivial.

Thus, the Hurwitz action may conjugate the elements g_0' at the beginning of the tuple by any element in the group generated by $\tilde{h}_1, \dots, \tilde{h}_{n\xi-d}$, which is equal to G. Thus, we may establish g_0 as the first entry. $\square$

We are now ready to conclude:

Theorem 6.20. *Let G be a finite non-abelian group which can be invariably generated by $c = (c_1, \dots, c_t)$, and let $\underline{\xi} = (\xi_1, \dots, \xi_t) \in \mathbb{N}^t$. Let $U \in R = R^{\mathbb{Z},c}_{G,\underline{\xi}}$ be a Hurwitz vector with $\partial U = 1$. Then for any $p \geq 0$, there are isomorphisms*

$$H_p(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; \mathbb{Z}) \cong H_p(\mathrm{Hur}^c_{G,(n+1)\cdot\underline{\xi}}; \mathbb{Z})$$

in the range $n > (8D_R(U) + \deg U)p + 7D_R(U) + \deg U$. For $b = b_0(\mathrm{Hur}^c_{G,(D_R(U)+1)\cdot\underline{\xi}})$, in the same range, we have

$$H_p(\mathrm{Hur}^c_{n\cdot\underline{\xi}}; \mathbb{Q}) \cong H_p(\mathrm{Conf}_{n\cdot\underline{\xi}}; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}^b\,.$$

Proof. The last statement of Proposition 6.19 tells us that the assumptions of Theorem 5.7 are satisfied for U. Indeed, for $p \geq 0$, we have

$$H_p(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; \mathbb{Z}) \cong H_p(\mathrm{Hur}^c_{G,(n+\deg U)\cdot\underline{\xi}}; \mathbb{Z})$$

as long as $n > (8D_R(U) + \deg U)p + 7D_R(U) + \deg U$.

Any $V = V(\underline{a}) \in R$ is also a $\mathbb{Z}$-stabilizing element for R by Proposition 6.19 and the fact that $\partial V = 1$. From the same proposition, we obtain that there is an $N \in \mathbb{N}$ such that for any $g \in \mathbf{c}$, $V^N \in R$ is equivalent under the Hurwitz action to a tuple whose first ξ entries equal g. Thus, for n in the stable range for V^N, multiplication by V^N factors into injective maps

$$H_p(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; \mathbb{Z}) \to H_p(\mathrm{Hur}^c_{G,(n+1)\cdot\underline{\xi}}; \mathbb{Z}).$$

Since in this range, multiplication by V^N is an isomorphism, all maps in the factorization must be isomorphisms as well.

Now, for any $n > (8D_R(U) + \deg U)p + 7D_R(U) + \deg U$, there is a $k \in \mathbb{N}$ such that $n + k \deg U$ is in the stable range for V^N. We finally obtain

$$\begin{aligned} H_p(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}; \mathbb{Z}) &\cong H_p(\mathrm{Hur}^c_{G,(n+k\deg U)\cdot\underline{\xi}}; \mathbb{Z}) \\ &\cong H_p(\mathrm{Hur}^c_{G,(n+k\deg U+1)\cdot\underline{\xi}}; \mathbb{Z}) \\ &\cong H_p(\mathrm{Hur}^c_{G,(n+1)\cdot\underline{\xi}}; \mathbb{Z}). \end{aligned}$$

By definition of $D_R(U)$, the number $b = b_0(\mathrm{Hur}^c_{G,n\cdot\underline{\xi}})$ of connected components is stable for $n > D_R(U)$. Fix $p \geq 0$ and $n > (8D_R(U) + \deg U)p + 7D_R(U) + \deg U$. Now, n is *always* in the stable range for the rational homology of $\{\mathrm{Conf}_{n\cdot\underline{\xi}} \mid n \geq 0\}$, given by $n \geq \min\{\frac{2p}{\min \underline{\xi}}, \frac{4p+\xi}{\max \underline{\xi}} - 1\}$ (Theorem 2.23 and Proposition 2.24). Indeed, we have $D_R(U) > 0$ because G is non-abelian by assumption (cf. Lemma 5.4).

We choose $k \geq 0$ such that $n + k \deg U$ is in the stable range for the stabilizing element V. We obtain

$$\begin{aligned} H_p(\mathrm{Hur}^c_{n\cdot\underline{\xi}}; \mathbb{Q}) &\overset{5.7}{\cong} H_p(\mathrm{Hur}^c_{(n+k\deg U)\cdot\underline{\xi}}; \mathbb{Q}) \\ &\overset{6.14}{\cong} H_p(\mathrm{Conf}_{(n+k\deg U)\cdot\underline{\xi}}; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}^b \\ &\underset{2.24}{\overset{2.23}{\cong}} H_p(\mathrm{Conf}_{n\cdot\underline{\xi}}; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}^b, \end{aligned}$$

which yields the remaining assertion. □

Example 6.21. We continue Example 6.5. Let $G = \mathcal{S}_3$. For $\underline{\xi} = (1, 1)$, the element $U = ((12), (123), (12), (123))$ satisfies $\partial U = 1$. Thus, $\deg U = D_R(U) = 2$. For any $p \geq 0$, we obtain $H_p(\mathrm{Hur}^c_{\mathcal{S}_3,(n,n)}; \mathbb{Q}) \cong H_p(\mathrm{Hur}^c_{\mathcal{S}_3,(n+1,n+1)}; \mathbb{Q}) \cong H_p(\mathrm{Conf}_{n,n}; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}^3$ in the stable range $n > 18p + 16$.

6.3. Outlook

Count of components Proposition 6.19 shows that the zeroth Betti number of Hurwitz spaces $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ which parametrize only connected covers is eventually stable for $n \gg 0$. This does not tell us *how many* connected components there are.

We might ask:

> For fixed G, c, and $\underline{\mu}$, are there invariants different from the boundary monodromy which separate connected components of $\mathrm{Hur}^c_{G,\underline{\mu}}$? What does a fine invariant system look like?

Now, assume that we know such a fine invariant system: If the invariants are computable, we know the (number of) connected components of $\mathrm{Hur}^c_{G,n\cdot\underline{\xi}}$ and are therefore able to answer the question whether the zeroth Betti number of the sequence $\{\mathrm{Hur}^c_{G,n\cdot\underline{\xi}} \mid n \geq 0\}$ stabilizes (if we did not already know it). If the answer is positive, the fine invariants might help us find a suitable element U, and if such a one exists, determine $D_R(U)$.

If c contains all nontrivial conjugacy classes in G, a theorem from an unpublished preprint by Conway and Parker ([CP88]) tells us that under an additional condition on G, the boundary monodromy will eventually become a fine invariant for the connected components of $\mathrm{Hur}_{G,n\cdot\underline{\xi}}$, once $n \gg 0$.

We took the following definition from [MM99]:

Definition 6.22. Let G be a finite group and F the free group over the elements of G. Let furthermore $R \subset F$ be the kernel of the canonical epimorphism $F \to G$. The *Schur multiplier* of G is defined as the group $M(G) = ([F,F] \cap R)/[F,R]$,

Theorem 6.23 (Conway–Parker, [CP88]; Fried–Völklein, [FV91])
Let G be a finite group whose Schur multiplier $M(G)$ is generated by commutators and $c = (c_1, \ldots, c_t)$ a list of all nontrivial conjugacy classes in G. Then there exists an $N \in \mathbb{N}$ with the following property: If $\underline{g}$ is a Hurwitz vector whose shape vector $\underline{\mu} = (n_{c_1}(\underline{g}), \ldots, n_{c_t}(\underline{g}))$ satisfies $\mu_i \coloneqq n_{c_i}(\underline{g}) \geq N$ for all $i = 1, \ldots, t$, then $\underline{g}$ is equivalent under the Hurwitz action of $\mathrm{Br}_{\underline{\mu}}$ to all Hurwitz vectors $\underline{h} \in c_1^{\mu_1} \times \ldots \times c_t^{\mu_t}$ with $\partial \underline{h} = \partial \underline{g}$.

From the previous example $G = \mathcal{S}_3$, we generalize to the dihedral groups D_k for $k \geq 3$. A couple of attempts have been made to calculate the number of Hurwitz orbits in D_k^n. In [CLP11], it is shown that the spherical braid group $\mathrm{Br}_n(S^2)$ acts transitively on Hurwitz vectors $\underline{g}$ with fixed *numerical type* and boundary $\partial \underline{g} = 1$, up to the diagonal action of $\mathrm{Aut}(\mathrm{D}_k)$. In many cases, the numerical type is just given by the Nielsen class; a precise definition of the numerical type can be found in the original article. This result corresponds to the irreducibility of the space of dihedral branched covers of $\mathbb{P}^1_{\mathbb{C}}$ of fixed numerical type.

A complete description of the Br_n-orbits in D_k^n can be found in [Sia09]. It is yet to be checked for what choices of Nielsen classes $\underline{\xi} \in \mathbb{N}_0^t$ the zeroth Betti number of $\mathrm{Hur}^c_{\mathrm{D}_k, n \cdot \underline{\xi}}$ indeed stabilizes, where c is a list of the nontrivial conjugacy classes in D_k. In the same article, fine invariants for the connected components of Hurwitz spaces can also be found for dicyclic and semidihedral groups. For symmetric groups $\mathcal{S}_k$, the determination of Hurwitz orbits in $\mathcal{S}_k^n$ is tackled in [BIT03].

Improvement of the stable range We note that the slope of the stable ranges for the homological stability of Hurwitz spaces deduced in this thesis are somewhat worse than the slopes in many other homological stability theorems, cf. also Chapter 2. It is not clear whether this is caused by the structure of the problem or due to the suboptimal proof. We ask:

> What is the optimal stable range for the homological stability theorems in this thesis?

Affine diagonals and further directions By combining the insights from Example 6.5, Example 6.21, and Theorem 6.13, we may construct further isomorphisms. For example, for any $p \geq 0$ and $n \geq 0$ odd, we have

$$H_p(\mathrm{Hur}^c_{\mathcal{S}_3,(n,n+1)};\mathbb{Z}) \cong H_p(\mathrm{Hur}^c_{\mathcal{S}_3,(n+2,n+3)};\mathbb{Z})$$

with stable range $n > 35p + 31$. This could be labeled *stabilization in the affine diagonal direction*. From the results of this thesis, we are not yet able to fill the whole (n_2, n_3)-lattice with such isomorphisms.

By Theorem 2.23, for any $\underline{\mu} \in \mathbb{N}^t$, $\underline{\xi} \in \mathbb{N}_0^t$, and $p \geq 0$, there is an isomorphism $H_p(\mathrm{Conf}_{\underline{\mu}};\mathbb{Z}) \cong H_p(\mathrm{Conf}_{\underline{\mu}+\underline{\xi}};\mathbb{Z})$ in the stable range $n \geq \frac{2p}{\min \underline{\mu}}$. Ideally, there would be an analogous result for Hurwitz spaces.

Thence, an initial question for further research in this direction would be inspired by Theorem 2.23, Theorem 6.13, and Theorem 6.14.

> Let G be invariably generated by c, and let $\underline{\mu} \in \mathbb{N}^t$. Is there a suitable function $r(p)$ such that for any $p \geq 0$, we have $H_p(\mathrm{Hur}^c_{G,n\cdot\underline{\mu}}) \cong H_p(\mathrm{Hur}^c_{G,n\cdot\underline{\mu}+\underline{\xi}})$ in the stable range $n \geq r(p)$?

Higher genus and genus stabilization In this thesis, we deal exclusively with covers of genus zero surfaces. With Harer's homological stability theorem for moduli spaces of curves (Theorem 2.25) in mind, it is a natural question whether homological stabilization happens not only in the direction of the number of branch points, but also in the direction of the genus of the covered surface.

Starting with the zeroth Betti number, this has been tackled in the articles [CLP15] and [CLP16] in a slightly different setting. The authors define a new topological invariant ϵ on the substratum $\mathcal{M}_g(G)$ of $\mathcal{M}_g$ which contains the algebraic curves admitting a faithful action by a fixed finite group G. It is shown that ϵ is a fine invariant for the connected components of $\mathcal{M}_g(G)$ for g sufficiently large.

Let $S_{g,r}$ be a Riemann surface of genus g with r boundary components. If we write $\mathrm{Hur}^c_{G,\underline{\mu}}(S_{g,r})$ for the higher-genus version of $\mathrm{Hur}^c_{G,\underline{\mu}} = \mathrm{Hur}^c_{G,\underline{\mu}}(D)$, we ask:

> Can we construct a map $\mathrm{Hur}^c_{G,\underline{\mu}}(S_{g,1}) \to \mathrm{Hur}^c_{G,\underline{\mu}}(S_{g+1,1})$ which is induced by the inclusion $S_{g,1} \hookrightarrow S_{g+1,1}$ given by gluing a torus with two boundary components to $S_{g,1}$? Does this map induce homological stability? Is there also a version for closed surfaces?

A. Simplicial Complexes

In this appendix, we review the most essential definitions for simplicial complexes and quote the results which are needed in this thesis, especially in Chapter 4. Main references are [Hud69, Ch. 1] and [Spa66, Ch. 3].

Definition A.1. A *simplicial complex* $\mathcal{K}$ is a set of non-empty finite sets that is closed under taking non-empty subsets. The singletons in $\mathcal{K}$ are called *vertices*, and the sets of cardinality $p+1$ are called *p-simplices*. A subset of a simplex $\alpha \in \mathcal{K}$ is called a *face*. A face of a p-simplex which is a $(p-1)$-simplex is called a *facet*.

Definition A.2. Let $\mathcal{K}$ be a simplicial complex.

(i) A set $\mathcal{K}' \subset \mathcal{K}$ is a *subcomplex* if it is a simplicial complex.

(ii) The set of p-simplices is denoted by $\mathcal{K}_p$, i.e., $\mathcal{K}_p = \{\alpha \in \mathcal{K} \mid |\alpha| = p+1\}$. For $p \geq 0$, the subcomplex $\mathcal{K}_{(p)} = \bigcup_{0 \leq i \leq p} \mathcal{K}_i \subset \mathcal{K}$ is called the *p-skeleton* of $\mathcal{K}$.

(iii) The *dimension* of $\mathcal{K}$ is the largest integer p such that $\mathcal{K}_p$ is non-empty, and infinite if no such number exists.

(iv) Let $\alpha \in \mathcal{K}$ be a simplex. The *(closed) star* of α is the subcomplex of $\mathcal{K}$ defined by $\mathrm{Star}(\alpha) = \{\beta \in \mathcal{K} \mid \alpha \cup \beta \in \mathcal{K}\}$. The *link* of α is the subcomplex defined by $\mathrm{Link}(\alpha) = \{\beta \in \mathrm{Star}(\alpha) \mid \alpha \cap \beta = \varnothing\}$.

(v) If $\mathcal{L}, \mathcal{L}'$ are simplicial complexes, their *join* $\mathcal{L} * \mathcal{L}'$ is defined as the simplicial complex $\mathcal{L} * \mathcal{L}' = \mathcal{L} \sqcup \mathcal{L}' \sqcup \{\alpha \sqcup \beta \mid \alpha \in \mathcal{L}, \beta \in \mathcal{L}'\}$. Its simplices are of the form $\alpha * \beta$, where $\alpha \in \mathcal{L} \cup \{\varnothing\}$, $\beta \in \mathcal{L}' \cup \{\varnothing\}$.

(vi) A *simplicial map* between simplicial complexes $\mathcal{K}$, $\mathcal{L}$ is a function $f\colon \mathcal{K}_0 \to \mathcal{L}_0$ such that for each simplex $\sigma \in \mathcal{K}$, we have $f(\sigma) := f(\{v \mid v \in \sigma\}) \in \mathcal{L}$.

(vii) A simplicial map $f\colon \mathcal{K}_0 \to \mathcal{L}_0$ is an *isomorphism of simplicial complexes* if it is a bijection and for all $\tau \in \mathcal{L}$, we have $f^{-1}(\tau) \in \mathcal{K}$.

Definition A.3. An *ordered simplicial complex* is a simplicial complex $\mathcal{K}$ together with a partial order $<$ on its set of vertices such that the restriction of $<$ to the vertices of any simplex in $\mathcal{K}$ is a total order. For a p-simplex $\alpha \in \mathcal{K}$ with set of vertices $\{v_0, \ldots, v_p\}$ such that $v_{i-1} < v_i$ for $i = 1, \ldots, p$, we write $\alpha = \langle v_0, \ldots, v_p \rangle$. Note that any simplicial complex may be given the structure of an ordered simplicial complex by imposing some total order on its set of vertices.[1]

On an ordered simplicial complex $\mathcal{K}$, we may define *face maps* $\partial_i \colon \mathcal{K}_p \to \mathcal{K}_{p-1}$ given by $\partial_i(\langle v_0, \ldots, v_p \rangle) = \langle v_0, \ldots, \hat{v}_i, \ldots, v_p \rangle$ for $i = 0, \ldots, p$.

Definition A.4. Let $\mathcal{K}$ be a simplicial complex.

(i) The *geometric realization* $|\mathcal{K}|$ of $\mathcal{K}$ is the set of all functions $\varphi \colon \mathcal{K}_0 \to I$ such that (i) the support of φ is a simplex and (ii) $\sum_{v \in \mathcal{K}_0} \varphi(v) = 1$. Topologized by the *weak topology*, it is a normal Hausdorff space. Every simplicial map f defines a map of geometric realizations which is continuous; this map is called the *geometric realization* of f.[2]

(ii) Let $\mathcal{K}$ be an ordered simplicial complex. The *barycentric coordinates* on its geometric realization $|\mathcal{K}|$ are defined as follows: For $\varphi \colon \mathcal{K}_0 \to I$ supported on (a subsimplex of) $\alpha = \langle v_0, \ldots, v_p \rangle \in \mathcal{K}$, we write $\varphi = [\alpha, (\varphi(v_0), \ldots, \varphi(v_p))]$.

If the simplicial complex is not ordered, the choice of an order on the vertices of every simplex enables us to use barycentric coordinates also in this case.

Lemma A.5. *Let $\mathcal{K}$ be a simplicial complex. For every vertex $v \in \mathcal{K}_0$, the geometric realization $|\operatorname{Star}(v)|$ of its star is contractible.*

Proposition A.6. *[Hat02, Thm. 2C.1] Let $\mathcal{K}$ be a finite simplicial complex and $\mathcal{L}$ an arbitrary simplicial complex. Let $\tilde{f} \colon |\mathcal{K}| \to |\mathcal{L}|$ be a continuous map. Then for some iterated barycentric subdivision $\mathcal{K}'$ of $\mathcal{K}$, there is a simplicial map $f \colon \mathcal{K}' \to \mathcal{L}$ such that the geometric realization of f is homotopic to $\tilde{f}$.*[3]

Definition A.7.

(i) A *combinatorial n-manifold* is a simplicial complex $\mathcal{K}$ such that for all p-simplices $\alpha \in \mathcal{K}_p$, the geometric realization $|\operatorname{Link}(\sigma)|$ is homeomorphic to either a sphere S^{n-p-1} or a ball D^{n-p-1}.

[1] This holds as long as we accept the ordering axiom O, which is strictly weaker than the axiom of choice, cf. [Gon95].

[2] For the definition of the weak topology, the proof that the realization is a normal Hausdorff space, and the construction of the map, cf. [Spa66, Ch. 3.1].

[3] For the definition of barycentric subdivisions, cf. [Spa66, Ch. 3.1].

(ii) If $\mathcal{K}$ is a combinatorial manifold, its geometric realization $|\mathcal{K}|$ is said to have the structure of a *piecewise linear (PL) manifold*.[4] If X is a topological space which is homeomorphic to the geometric realization of a combinatorial manifold $\mathcal{K}$, we say that $\mathcal{K}$ is a *triangulation* of X.

(iii) The *boundary* of a combinatorial n-manifold $\mathcal{K}$ is the smallest subcomplex of $\mathcal{K}$ which contains all simplices $\alpha \in \mathcal{K}$ such that $|\operatorname{Link}(\alpha)|$ is homeomorphic to a ball.

Definition A.8.

(i) A *semi-simplicial set* $\mathbf{O} = \bigsqcup_{q \geq 0} \mathbf{O}_q$ is the disjoint union of sets $\mathbf{O}_q$, called the *set of q-simplices* in $\mathbf{O}$, together with *face maps* $\partial_i = \partial_i^{(q)} \colon \mathbf{O}_q \to \mathbf{O}_{q-1}$ such that for $i < j$, we have $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$.

(ii) A *semi-simplicial space* is a semi-simplicial set $\mathbf{O}$ which is a topological space together with a collection of subspaces $\mathbf{O}_q$ and continuous face maps ∂_i.

(iii) Let $\mathbf{O}, \mathbf{O}'$ be semi-simplicial sets (spaces) with respective face maps ∂_i, ∂_i'. A *semi-simplicial isomorphism* is a bijection (homeomorphism) $\psi \colon \mathbf{O} \to \mathbf{O}'$ whose restriction to $\mathbf{O}_q$ yields a bijection (homeomorphism) $\psi|_{\mathbf{O}_q} \colon \mathbf{O}_q \to \mathbf{O}'_q$ for all $q \geq 0$, and such that for all $q \geq 0$ and $i = 1, \dots, q$, $\partial_i' \circ \psi = \psi \circ \partial_i \colon \mathbf{O}_q \to \mathbf{O}'_{q-1}$.

From the definition, it is clear that any ordered simplicial complex has the structure of a semi-simplicial set. The geometric realization of an ordered simplicial complex is a semi-simplicial space, cf. also Remark 5.13.

[4] In fact, the definition of a PL manifold is more general. We only need it in the context of geometric realizations of combinatorial manifolds, though. For the general definition, cf. [Hud69, Ch. 1.5].

B. Spectral Sequences

Semi-simplicial sets Let $\mathcal{O}$ be a semi-simplicial space with face maps ∂_i and $\mathcal{O}_q$ its set of q-simplices. Now, $\mathcal{O}$ is filtered by the semi-simplicial structure on $\mathcal{O}$. In this case, the spectral sequence associated to a filtered space, cf. [FFG86, §18], reads as

$$E^1_{qp} = H_p(\mathcal{O}_p; \mathbb{Z}) \Longrightarrow H_{p+q}(\mathcal{O}; \mathbb{Z}) \tag{B.1}$$

and calculates the homology of the total complex. The differentials on the E^1-page are given by the alternating sum $\mathrm{d} = \sum_{i=0}^{q} (-1)^i \partial_{i*} \colon E^1_{qp} \to E^1_{q-1,p}$.

Hyperhomology Let R be a ring and M a left R-module. Let furthermore $A_\bullet$ be a chain complex of flat right R-modules which is bounded from below. By [Wei94, Appl. 5.7.8], there are two associated spectral sequences:

$$E^1_{pq} = \mathrm{Tor}^R_q(A_p, M) \Longrightarrow H_{p+q}(A_\bullet \otimes M), \tag{B.2}$$

$$E^2_{pq} = \mathrm{Tor}^R_p(H_q(A_\bullet), M) \Longrightarrow H_{p+q}(A_\bullet \otimes M). \tag{B.3}$$

The spectral sequence (B.3) is also called the *universal coefficients spectral sequence*. If in (B.2) we let $A_\bullet$ be a free resolution of a right R-module $\bar{R}$, we obtain

$$E^1_{pq} = \mathrm{Tor}^R_q(A_p, M) \Longrightarrow \mathrm{Tor}^R_{p+q}(\bar{R}, M). \tag{B.4}$$

Base change for Tor Let $R, \bar{R}$ be rings and $p \colon R \to \bar{R}$ a ring homomorphism. If A is a right $\bar{R}$-module and M a left R-module, the Grothendieck spectral sequence yields the existence of a spectral sequence

$$E^2_{pq} = \mathrm{Tor}^{\bar{R}}_p(A, \mathrm{Tor}^R_q(\bar{R}, M)) \Longrightarrow \mathrm{Tor}^R_{p+q}(A, M) \tag{B.5}$$

cf. [Wei94, Cor. 5.8.4 & Appl. 5.8.5].

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